An Infinitely Farsighted Stable Set

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Abstract

In this paper, we first introduce and characterize a new and general concept of a credible deviation in which a coalition may not only deviate from a deviant coalition, but may also merge with some residual players. Thus objections are not necessarily nested in the sense of coming from subsets of progressively smaller coalitions. We then motivate and introduce an infinitely farsighted stable set which is not amenable to similar criticisms as a traditional von Neumann-Morgenstern stable set or a Harsanyi stable set. After noting its general properties, we prove existence and characterize infinitely farsighted stable sets for general three-player superadditive characteristic function games with empty or nonempty cores as well as for convex games with any number of players.

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1. Introduction

In cooperative game theory, the von Neumann-Morgenstern (vNM) stable sets (von Neumann and Morgenstern, 1944) as a solution concept have been dominated by the core introduced to the literature almost a decade later by Gillies (1953), at least judging by their applications to mainstream economics. This is regrettable because the vNM stable sets, unlike the core, seem to be a promising (and perhaps the only) solution concept for games with empty cores, since they are generally larger than the core and can be nonempty even if the core is empty. However, with the publication of Ray and Vohra (2014), interest in stable sets seems to have revived. Following Harsanyi's (1974) critique and his modification of the vNM stable sets, referred to as Harsanyi stable sets, Ray and Vohra (2014) propose a modification of Harsanyi stable sets, and thereby of the original vNM stable sets, which beautifully restores feasibility and coalitional sovereignty ignored implicitly in the definition of Harsanyi stable sets. However, like traditional vNM stable sets, both Harsanyi and Ray and Vohra stable sets, called the farsighted stable sets, have one feature in common, namely that, coalitional deviations may not be "credible".

In this paper, we first introduce and characterize a new and general concept of a credible deviation in which a coalition may not only deviate from a deviant coalition, but may also merge with some residual players. Thus, unlike Bernheim, Peleg, and Whinston (1987) and Ray (1989), "objections" are not necessarily nested in the sense of coming from subsets of progressively smaller coalitions.[1] According to this notion, a deviation (defined in the space of imputations) is credible if it is immune to every deviation which is immune to every deviation … *ad infinitum*. Since a credible deviation, by definition, is immune to every credible deviation, the credible deviations form a stable set in the sense that a deviation belongs to the set if and only if it is immune to every deviation in the set and any deviation not in the set is not immune to every deviation in the set.[2] Conversely, *any* stable set of *deviations* is a set of credible deviations, since, by definition of a stable set, every deviation in the set is immune to every other deviation in the set and no deviation outside the stable set is immune to every deviation in the set. Thus, in what follows, a stable set of deviations and a stable set of *credible* deviations mean the same thing.

[1] See e.g. Bernheim et al. (1987, fn. 2) on the generality and difficulty of dealing with this type of deviations.
[2] See Section 3.1and definitions 2 and 3 below for the formal definitions of a deviation, an imputation immune to a deviation, and a credible deviation.

Given a stable set of deviations and, therefore, a stable set of credible deviations, we define from it a set of imputations, which is larger than the stable set of deviations, and show that it is stable in the sense to be made clear below. This stable set is the set of all imputations which are immune to every deviation in the stable set of deviations and, therefore, it includes the stable set of deviations itself. Since credible deviations, by definition, are infinitely farsighted, we name it an infinitely farsighted stable set (IFSS).

A so-defined IFSS is not amenable to similar criticisms as a vNM stable set (see Harsanyi, 1974) or a Harsanyi stable set (see Ray and Vohra, 2014, p.2-3), since (infinite) farsightedness is already built into the definition and it respects both feasibility and coalitional sovereignty in exactly the same way as does a traditional vNM stable set. Recall that a set of imputations V is a vNM stable set if every imputation is either a member of V or dominated by a member of V, but not both. Thus, a vNM stable set may be interpreted as a "standard of behavior", i.e., a set of "conventional outcomes" which are given a chance to dominate any proposal that might be put forward during pre-play negotiations.

In contrast, an IFSS is a stable set of imputations which are not dominated by a set of credible deviations. But, as will be shown, the IFSSs and the vNM stable sets, though motivated and defined quite differently, are closely related in that a vNM stable set may be a subset of an IFSS. In fact, that is indeed so if a vNM stable set, like an IFSS, also contains a stable set of deviations. But, as an example shows, a vNM stable set may not always contain a stable set of deviations and, thus, may not necessarily be a subset of an IFSS.

Every IFSS of a transferable utility characteristic function (TUCF) game, like every traditional vNM stable set, contains the core, but not generally equal to the core. This leads to an alternative interpretation of the core, namely that, the core imputations are not dominated by the credible deviations and those on the boundary of the core are credible deviations themselves. Thus, (infinite) farsightedness is implicit in the concept of core, even though it is defined by the property of an imputation of not being "myopically blocked". However, as will be shown, not every credible deviation is a core imputation and an IFSS is generally larger than a stable set of deviations, especially if the core of the game has a nonempty interior. Similarly, farsightedness is also implicit in those vNM stable sets – though defined again by a myopic notion of dominance -- which contain a stable set of deviations and, thus, are contained in an IFSS.

Existence of stable sets is usually difficult to prove. Lucas (1968) shows that a vNM stable set may not exist in superadditive TUCF games with nonempty cores. However, Ray and Vohra (2014) show that in games with *nonempty* cores, multiple farsighted stable sets exist, each of which consists of a single imputation which belongs to the core and each imputation in the interior of the core by itself forms a farsighted stable set. They also prove existence of farsighted stable sets in proper simple games with *empty* cores. But other than this exceptional result for proper simple games, not much else is known about the existence of stable sets in general TUCF games with empty cores.

In this paper, we prove existence of an IFSS for general three-player superadditive TUCF games with empty or nonempty cores. As will be shown, IFSSs generally consist of many imputations and include for each coalition at least one imputation which is "maximally feasible" for it. In order to contrast IFSSs with other farsighted solution concepts, we show that in games with empty cores, *every* "interior" imputation is a Harsanyi stable set and the largest consistent set (Chwe, 1994) includes at least all interior imputations.[3] Thus, in contrast to IFSSs, these farsighted solution concepts do not have much predictive power in games with empty cores.

The contents of this paper are as follows. In Section 2, we discuss a well-known example to motivate and introduce the concept of a credible deviation. In section 3, we motivate and introduce the concept of an IFSS. This Section also contrasts IFSSs with other concepts of farsighted stable sets. In Section 4, we prove existence and comprehensively characterize IFSSs for general three-player superadditive TUCF games with empty or nonempty cores as well for convex games with any number of players. In Section 5, we draw the conclusion.

2. A motivating example

[3] These results regarding the existence and characterization of Harsanyi stable sets and the largest consistent set for games with *empty* cores complement those for games with *nonempty* cores (Chwe, 1994 and Bèal et al., 2008).

We begin with a motivating example which has two firms, labelled 1 and 2, emitting smoke which reduces the profits of a neighborhood laundry, labelled 3.[4] The situation is described by a characteristic function v such that:[5]

$$v(1) = 3000,\ v(2) = 8000,\ v(3) = 24000$$

$$v(12) = 15000,\ v(13) = 31000,\ v(23) = 36000$$

$$v(123) = 40000$$

Clearly, the characteristic function v is (strictly) superadditive which implies that the grand coalition, where the externalities are fully internalized, is the *unique* efficient coalition. Thus, efficiency can be achieved *only if* the grand coalition forms, i.e., all three firms merge. The core of this characteristic function game is empty, since, by definition, a feasible payoff vector (x_1, x_2, x_3) belongs to the core only if $x_i + x_j \geq v(ij)$ for all $i, j \in \{1, 2, 3\}$.[6] But there is no feasible payoff vector which satisfies these inequalities, since they imply $x_1 + x_2 + x_3 \geq \frac{1}{2}[v(ij) + v(ik) + v(jk)] = 41000 > v(123)$.

The argument that has been used against the stability of the grand coalition runs as follows: Consider any three-party agreement between the firms, say (5250, 9750, 25000). Since 2 and 3 would be getting in total \$34,750 but could obtain \$36,000 if they made an agreement between themselves in which, e.g., 2 receives \$10,000 and 3 receives \$26,000, this two-party agreement would clearly be preferred by both 2 and 3 rather than the three-party agreement (5250, 9750, 25000). However, this two-party agreement between 2 and 3 would not be a stable agreement either, since firm 1 would now be operating independently and earning \$3,000 and 2 would be earning \$10,000 (as a result of the agreement with 3), the total profit earned by 1 and 2 would be \$13,000.[7] But if 1 and 2 made an agreement between themselves, their joint profits would amount to \$15,000 -- which they would find profitable to

[4] This example has previously appeared in Aivazian and Callen (1981) among others to question validity of the Coase theorem in three-player superadditive TUCF games with empty cores. Also, see Koldstad (2000, Ch. 6) for a lucid discussion of this example.
[5] To economize on parenthesis and commas, we shall denote coalitions {1, 2}, {1, 3}, {2, 3}, and {1, 2, 3} simply by 12, 13, 23 and 123, respectively. The coalitional payoffs are measured in dollars.
[6] We shall follow the convention that $j = i + 1$ (mod3). E.g., if $i = 3$, then $j = 1$ and $k = 2$.
[7] If the two-party agreement between firms 2 and 3 is translated into an imputation as defined below, then firm 1 will have a payoff of \$4,000 instead of \$3,000 and the total payoff of firms 1 and 2 will be \$14,000 instead of \$13,000.

the previous situation. Thus, 2 would leave the two-party agreement with 3 and form a coalition with 1 and divide their joint profit of $15,000 as $4,000 for firm 1 and $11,000 for firm 2 such that they would each be better-off. But this is not the end of the story either. As Aivazian and Callen (1981) argue, the re-contracting process will continue endlessly and, therefore, the grand coalition is not stable.

One problem with the re-contracting process described above is that firm 3 at the time of breaking the initial three-party agreement (5250, 9750, 25000) and entering into a separate two-party agreement with firm 2 assumes that it would get a profit of $26,000 without taking into account the fact that firm 2 would subsequently break the agreement and form a coalition with firm 1 instead, leaving it alone to earn a profit of $24,000 -- less than the $25,000 it was getting in the initial three-party agreement before entering into the two-party agreement with firm 2.[8]

Another problem is that firms 2 and 3 are assumed to divide their $36,000 profit *arbitrarily* among them except that they should each be better-off compared to the initial agreement (5250, 9750, 25000) without any concern for the impact it will have on the stability of their coalition.

Also, notice that the re-contracting process assumes implicitly that a player may leave a two-member deviant coalition and merge with the residual player to form another two-member coalition. Deviations of this type are unconventional and have not been studied much previously. Most solution concepts in game theory which allow deviations from deviations (see e.g. Bernheim, Peleg, and Whinston, 1987 and Ray, 1989) do not allow mergers between coalitions deviating from a deviant coalition and residual players. Thus, deviations from deviations can only result in smaller coalitions, but not so if mergers with residual players are allowed and deviations and mergers can continue forever as seen above.

3. Credible deviations and infinitely farsighted stable sets

The purpose of this section is to motivate and introduce a notion of a credible deviation which does not rule out mergers between coalitions deviating from deviant coalitions and

[8] In other words, the firms are myopic about their prospects.

residual players and then introduce a concept of a stable set based on this notion. We need the following definitions.

A TUCF game is a pair (N, v) where $N = \{1,2,\dots,n\}, n \geq 3$, is the finite set of players and $v(S)$ is the transferable worth that members of coalition S will have to divide among themselves if they were to cooperate together and with no one outside S. A TUCF game (N, v) is superadditive if $v(S \cup T) \geq v(S) + v(T)$ for all disjoint coalitions S and T.

A payoff vector $x = (x_1, \dots, x_n)$ is *feasible* if $\sum_{i\in N} x_i = v(N)$. In words, a feasible payoff vector represents a division of the worth of the grand coalition, and is efficient, i.e. maximizes the total payoff of the players, if the game is superadditive. An *imputation* of (N, v) is a feasible payoff vector x such that $x_i \geq v(i)$ for each $i \in N$. Let X denote the set of all imputations of (N, v). An imputation x is *dominated* by an imputation y if there is a coalition S such that $\sum_{i\in S} y_i \leq v(S)$ and $y_i > x_i$ for all $i \in S$. Let $\succ$ denote this dominance relation, that is, for any two imputations x and y, $y \succ x$ if there is a coalition S such that $\sum_{i\in S} y_i \leq v(S)$ and $y_i > x_i$ for all $i \in S$. An imputation y is *feasible* for S if $\sum_{i\in S} y_i \leq v(S)$ and *maximally feasible* for S if $\sum_{i\in S} y_i = v(S)$. For easy reference, we reproduce here the definition of a vNM stable set (see e.g. Osborne and Rubinstein, 1994).

Definition 1 Given a TUCF game (N, v), a subset Y of the set of imputations X is a vNM stable set if it satisfies the following two conditions.

Internal stability: If $x \in Y$, then there is no $y \in Y$ such that $y \succ x$.
External stability: If $x \in X\backslash Y$, then there is a $y \in Y$ such that $y \succ x$.

The definition does not rule out that an imputation y in a vNM stable set may be such that $\sum_{i\in T} y_i < v(T)$ for some coalition T.[9] A vNM stable set in the game in Example 2 below indeed includes such imputations. This is odd, since internal stability then implies that imputations in a vNM stable set should not be dominated even by an imputation y such that $\sum_{i\in T} y_i < v(T)$ for some coalition T, though T itself has no incentive to choose y, since it can *unilaterally* choose instead an alternative imputation z such that $\sum_{i\in S} z_i = v(T)$ and $z_i > y_i$ for all $i \in T$. All the more so, since, by definition of a vNM stable set, the coalitions

[9] In contrast, the core includes no such imputation.

are supposedly myopic and care only about their immediate payoffs. But it can be justifiably argued that such an imputation y may belong to the vNM stable set not because of coalition T, but because it is the choice of some other coalition $S \neq T$ for which it is maximally feasible. If that is so, for every imputation y in a stable set such that $\sum_{i \in T} y_i < v(T)$ for some coalition T there must be at least some coalition S $(\neq N, T)$ for which $\sum_{i \in S} y_i = v(S)$.[10]

3.1 *A new and general definition of a credible deviation*

As is the convention in the related literature, we define a deviation in the space of imputations. More specifically, a *deviation* by a coalition T is an imputation y such that $\sum_{j \in T} y_j \leq v(T)$ and $\sum_{i \in S} y_i = v(S)$ for some coalition S $(\neq N)$ which is not necessarily the deviating coalition T.[11] That is, a deviation must be feasible for the deviating coalition and maximally feasible for *some* coalition, though not necessarily for the *deviating* coalition. It is worth noting that this definition allows for the possibilities that a deviation y by a coalition T is such that $\sum_{j \in T} y_j < v(T)$, but for some coalition $S \neq T$, we have $\sum_{i \in S} y_i = v(S)$ or $\sum_{j \in T} y_j = v(T)$, but for no coalition $S \neq T$, we have $\sum_{i \in S} y_i = v(S)$. However, for the reasons discussed in the preceding paragraph, the definition rules out deviations y by a coalition T such that $\sum_{j \in T} y_j < v(T)$ and there is no other coalition S for which $\sum_{i \in S} y_i = v(S)$. That is, the deviations which are not maximally feasible either for the deviating coalition or for any other coalition are ruled out.

Definition 2 An imputation x is *immune* to a deviation y if there is no coalition S such that $\sum_{j \in S} y_j \leq v(S)$ and $y_i > x_i$ for all $i \in S$, i.e., x is not dominated by y.

Let D denote the set of all deviations. Then, $D \subset X$. Thus, the dominance relation in Definition 2 is the same as in the original definition of a vNM stable set, except that it is defined over the set $D \times X$ rather than $X \times X$.

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[10] Going further, one may postulate that a stable set must contain for *each* coalition S at least one imputation y such that $\sum_{i \in S} y_i \geq v(S)$ and a stable set that does not meet this requirement is "coercive". Fortunately, as will be seen below, this requirement is met by the IFSSs in all cases in which we are able to prove their existence. This requirement is met by the core, but in a rather strong way in that *every* core imputation y is such that $\sum_{i \in S} y_i \geq v(S)$ for *every* coalition S.

[11] By definition of an imputation, the equality is always satisfied for $S = N$.

Definition 3 A deviation is credible if it is immune to every deviation which is immune to every deviation … *ad infinitum*.

Since deviations belong to the space of imputations, the definition permits a subset of players to leave a current coalition, merge with some outside players, and choose an alternative deviation which is feasible for the coalition so-formed and maximally feasible for some coalition. Thus, "objections" need not come *only* from subsets of increasingly smaller coalitions and, as seen in the motivating example, can continue to come forever. [12]

Since a credible deviation, by definition, is immune to every credible deviation,[13] the credible deviations form a stable set in the sense that a deviation belongs to the set if and only if it is immune to every deviation in the set and any deviation not in the set is not immune to every deviation in the set. Conversely, *any* stable set of *deviations* is a set of credible deviations, since, by definition of a stable set, every deviation in the set is immune to every other deviation in the set and no deviation outside the stable set is immune to every deviation in the set. Thus, in what follows, a stable set of deviations and a stable set of *credible* deviations mean the same thing and a deviation is credible if and only if it belongs to a stable set of deviations.

Let Z denote a stable set of deviations. Then, Z is a stable set in the space of deviations, but not necessarily in the space of imputations X. However, as will be seen below, we can construct from Z another set Y which is a stable set in X in the sense to be made clear below. We now introduce a stable set which is identical to a vNM stable set (see Definition 1) except that the deviations are required to be credible.

Definition 4 Given a TUCF game (N, v), a subset Y of the set of imputations X is a credible stable set if it satisfies the following two conditions:

[12] Credibility of deviations or strategies is an important requirement in game theory. E.g., the concept of coalition-proof Nash equilibrium (Bernheim et al., 1989), in contrast to a strong Nash equilibrium (Aumann, 1959), requires blocking deviations to be "credible". Similarly, the concept of subgame-perfect Nash equilibrium in contrast to a Nash equilibrium, requires the strategies to be "credible". However, the notions of credibility differ across the literature. The one introduced presently is a new addition and is of an independent interest.

[13] This follows from the fact that the "chain" is required to be infinite. More precisely, if a deviation x^1 is immune to every deviation x^2 which is immune to every deviation x^3 and so on, then every x^2 satisfies exactly the same property as x^1, since the chain is infinite. Hence, if x^1 is credible, then so is every x^2 and so on.

Internal stability: If $x \in Y$, then there is no credible deviation $y \in Y$ such that $y \succ x$.
External stability: If $x \in X \setminus Y$, then there is a credible deviation $y \in Y$ such that $y \succ x$.

Since a deviation y is credible if and only if it belongs to a stable set of deviations, external stability of a credible stable set implies that it must contain a stable set of deviations and, by internal stability, no imputation in the credible stable set should be dominated by a deviation in this stable set of deviations, but, by external stability, every imputation not in the credible stable set should be dominated by some deviation in this stable set of deviations. In contrast, no imputation in a vNM stable set should be dominated by an *imputation* in the vNM set which is not necessarily a deviation but may only be an imputation which is feasible for some coalition and every imputation outside the vNM stable set should be dominated by an imputation which is not necessarily a deviation but is feasible for some coalition. Since every deviation, by definition, is an imputation which is feasible for some coalition, but not every imputation which is feasible for some coalition is a deviation, a vNM stable set could be a subset of a credible stable set. Proposition 2 below confirms this intuition and identifies the exact conditions under which that is indeed so.

3.2 *An infinitely farsighted stable set*

Definition 5 Given a TUCF game (N, v), a set of imputations Y is an infinitely farsighted stable set if there is a stable set of deviations Z such that Y is the set of all imputations which are not dominated by any deviation in Z.

Proposition 1 Given a TUCF game (N, v), if Y is an infinitely farsighted stable set, then Y is a credible stable set.

Proof: Since Y is an infinitely farsighted stable set, there is a stable set of deviations Z such that Y is the set of all imputations which are not dominated by any deviation in Z. Since Z is a stable set, if $x \in Z$, then there is no $y \in Z$ such that $y \succ x$. Since Y is the set of all imputations which are not dominated by deviations in Z, it follows that $Y \supset Z$. Since Y is the set of all imputations which are not dominated by any deviation in Z, if $x \in Y$, there is no $y \in Z$ such that $y \succ x$. Since Y is the set of all imputations which are not dominated by any deviation in Z, for any $x \in X \setminus Y$ there must be a $y \in Z$ such that $y \succ x$. This proves that for any $x \in Y$, there is no $y \in Z \subset Y$ such that $y \succ x$ and for any $x \in X \backslash Y$ there is a $y \in Z \subset Y$ such that $y \succ x$, The proof now follows by noting that a deviation y is credible if and only if it belongs to the set Z. ■

It is worth noting that no imputation in the set $Y \backslash Z$ is a deviation, since every deviation which is not dominated by any deviation in Z is itself a credible deviation and, thus, belongs to Z. Thus, the existence of a stable set of credible deviations is the key to the definition and existence of an IFSS. Since credible deviations, by definition, are infinitely farsighted, farsightedness is already built into the definition of an IFSS. Additional characterizations of the IFSSs follow from their relationship with the core and the vNM stable sets.

Recall that a feasible payoff vector x belongs to the core of (N, v) if $v(S) \leq \sum_{i \in S} x_i$ for every coalition S and in the interior of the core if $v(S) < \sum_{i \in S} x_i$ for every $S \neq N$ or on the boundary of the core if $\sum_{i \in S} x_i = v(S)$ for some coalition $S \neq N$. Since the core -- if nonempty -- has a nonempty boundary,[14] it includes at least some imputations which are deviations.

Proposition 2 Let (N, v) be a TUCF game. Then (1) no IFSS is a subset of another, (2) every IFSS contains the core and is strictly larger than a stable set of credible deviations if the core has a nonempty interior, and (3) every vNM stable set which contains a stable set of deviations is a subset of an IFSS.

Proof: (1) Suppose contrary to the assertion that Y and Y' are two IFSSs such that $Y \subset Y'$ and $Y \neq Y'$. Then there must be stable sets of deviations Z and Z' such that Y and Y' are the sets of all imputations which are not dominated by deviations in Z and Z', respectively.Since $Z \subset Y \subset Y'$ and Z is a stable set of credible deviations, no imputation in Y' must be dominated by a deviation in Z. But that contradicts that Y is the largest set of imputations which are not dominated by deviations in Z. Hence our supposition is wrong and Y cannot be a subset of Y'.

(2) Let Y be an IFSS. Since every member of the core is an imputation and no core imputation, by definition, is dominated by a deviation (credible or not), it follows that the

[14] The core is a closed (and convex) set, since, by definition, it satisfies a system of *weak* linear inequalities.

core is contained in Y, since Y, by definition, contains a set of deviations Z and is the set of all imputations which are not dominated by the deviations in Z. Let C denote the set of all core imputations. Then C consists of two separate parts: the set of core imputations which are deviations, i.e., $x \in C$ such that $\sum_{i \in S} x_i = v(S)$ for some coalition $S \neq N$ and the (possibly empty) set of core imputations which are not deviations and, therefore, are in the interior of the core, i.e., the set of imputations $x \in C$ such that $\sum_{i \in S} x_i > v(S)$ for every coalition $S \neq N$. Let C_1 and C_2 denote the first and the second parts, respectively. Then $C_1 \neq \emptyset$, since the core is nonempty and therefore it has a nonempty boundary. Furthermore, $C_1 \subset Z$, since $C_1 \subset Y$ and, therefore, each deviation in C_1 is immune to every credible deviation and, thus, itself a credible deviation. By definitions, $C_1 \cap C_2 = \emptyset$, $C_1 \cup C_2 = C$, $C_2 \subset Y$, and $C_2 \cap Z = \emptyset$. Since $Z \subset Y$, $C_2 \subset Y$, and $C_2 \cap Z = \emptyset$, it follows that Y is strictly larger than the set Z if $C_2 \neq \emptyset$, i.e., if the core has a nonempty interior.

(3) Let V be a vNM stable set such that there is a stable set of deviations $Z \subset V$. Then, by internal stability of V, for each $y \in V \setminus Z$, there is no $z \in Z$ such that $z \succ y$. Let Y denote the set of all imputations which are not dominated by the deviations in Z. Then, $Z \subset Y$, Y is an IFSS, and $V \setminus Z \subset Y \setminus Z$, since Y is the set of *all* imputations which are not dominated by deviations in Z. Hence $V \subset Y$ and Y is an IFSS. ■

Corollary If an infinitely farsighted stable set Y coincides with a stable set of deviations Z, then Y is a vNM stable set.

The proposition does not imply that an IFSS is generally equal to the core. Indeed, as will be shown, it can be nonempty even if the core is empty. However, the proposition does imply that if an IFSS exists, then it contains the core and each core imputation is immune to every credible deviation and every deviation that belongs to the core is itself a credible deviation. This might explain why the core, as observed by Ray and Vohra (2014), often exhibits very powerful farsighted stability properties, even though it is defined by the property of an imputation of not being myopically blocked. Similarly, farsightedness is also implicit in at least those vNM stable sets – though supposedly defined again by a myopic notion of dominance -- which contain a stable set of credible deviations and, thus, are contained in an

IFSS. In fact, as will be seen, in many cases a vNM stable set coincides with an IFSS, as the Corollary above claims.

3.3 *Other farsighted solution concepts*

Harsanyi (1974) seems to have been the first to motivate and introduce farsightedness in the context of stable sets. Since Harsanyi stable sets are known to include imputations that do not belong to the core, they also seem to be a promising solution concept for games with empty cores. Thus it may be worth contrasting them with the IFSSs, especially if the core is empty. We need the following definitions: A TUCF game (N, v) is *normalized* if $v(i) = 0, i = 1, \dots, n$. It is well-known that there is no loss of generality in restricting to games which are normalized. We also assume that $v(N) > 0$, i.e. the game is essential. Only essential games are of interest, since in normalized games which are not essential the only conceivable imputation is $x_i = v(i) = 0, i \in N$, and, thus, there is no need to study solution concepts for such games.

For any two imputations x and y, x is said to be *feasible from* y *via a coalition* S if $\sum_{i \in S} x_i \leq v(S)$, to be denoted by $y \rightarrow_S x$. Notice that $y \rightarrow_S x$ if and only if $z \rightarrow_S x$ for every $z \in X$. An imputation x *farsightedly dominates* an imputation y, to be denoted by $x \gg y$, if there exists a finite sequence of imputations $y = x^1, x^2, \dots, x^m = x$ and a sequence of coalitions $S_1, S_2, \dots . S_{m-1}$ such that for each $j = 1,2, \dots m-1$, $x^j \rightarrow_{S_j} x^{j+1}$and $x_i^j < x_i$ for each $i \in S_j$.[15]

Clearly, the dominance relation "$\gg$ " underlying the definition of a Harsanyi stable set is weaker than the dominance relation "$\succ$ " underlying the definitions of both vNM stable sets and the IFSSs in the sense that for any x and y, if $x \succ y$, then $x \gg y$, but the converse is not true. Thus, a Harsanyi stable set may be more exclusive. The following important result confirms this intuition.

[15] It is worth noting that the dominance chain $x^1, x^2, \dots, x^m$ may terminate at an imputation x which is not maximally feasible for any coalition, leave alone for the coalitions $S_j, j = 1, \dots, m-1$, which induce the chain. Thus, the "deviations" in the dominance chain may not be "credible".

Proposition 3 (Béal et al., 2008) Let (N, v) be a normalized TUCF game in which $v(T) > 0$ for some $T \subset N, T \neq N$.Then, a set of imputations is a Harsanyi stable set if and only if it contains a single imputation x such that for some coalition $S \subset N$, $\sum_{i \in S} x_i \leq v(S)$ and $x_i > 0$ for each $i \in S$.

In light of this result, Ray and Vohra (2014) note that if the core is nonempty, then no imputation in the interior of the core can be part of a Harsanyi stable set. Here we characterize the Harsanyi stable sets for games with empty cores. We need the following additional definition: An imputation x of a game (N, v) is called *interior* if $x_i > v(i), i = 1, \dots, n$. Notice that given an imputation of an essential game, there exists an interior imputation which is arbitrarily close to it.

Proposition 4 Let (N, v) be a normalized TUCF game with an empty core. Then, every set containing a single interior imputation is a Harsanyi stable set.

<u>Proof</u>:[16] Let x be an interior imputation. Clearly, the singleton set $\{x\}$ satisfies internal stability. We prove that it also satisfies external stability by showing that x farsightedly dominates every other imputation z, i.e., there exists a dominance chain from z to x. Since x is an interior imputation and the core of (N, v) is empty, $x_i > v(i) = 0$ for all i and $v(S) > \sum_{i \in S} x_i$ for some $S \neq N$. Furthermore, coalition S can neither be a singleton nor equal to N, since x is an imputation. Since x and z are both imputations $z_i < x_i$ for at least some player i. Three cases arise: (i) the player $i \in S$, (ii) the player $i \notin S$ and there is also a player $k \notin S, k \neq i$, and (iii) $S = N \backslash i$. The dominance chains in the three cases are defined as follows: (i) $z \rightarrow_{\{i\}} y \rightarrow_S x$, where $y_j = 0$, for all $j \in S$ including $j = i$, (ii) $z \rightarrow_{\{i\}} y \rightarrow_S x$, where $y_k = v(N)$ and $y_j = 0$ for all $j \neq k$ including i, and (iii) $z \rightarrow_{\{i\}} y \rightarrow_{\{k\}} y' \underset{S}{\rightarrow} x$, where $y_j = v(N)$ for some $j \in S, j \neq k$, $y_h = 0$ for all $h \neq j$ including k, $y'_i = v(N)$, and $y'_j = 0$ for all $j \in S$. It is easily seen that the final payoff of every member of every deviating coalition in each dominance chain is higher than its payoff at the time of the deviation. This proves that $\{x\}$ is also externally stable and, thus, a Harsanyi stable set. ■

[16] An alternative, but less instructive proof follows from Proposition 3 by noting that every interior imputation of a game with an empty core meets the required conditions of that proposition.

The proof is quite instructive. Coalition S is the deviating coalition at the last step in each of the three possible dominance chains and in each chain it chooses the imputation x which results in the *final* payoffs $x_i, i \in S$, such that $\sum_{i\in S} x_i < v(S)$ despite the fact that it could have chosen instead an imputation x' such that $\sum_{i\in S} x'_i = v(S)$ and $x'_i > x_i$ for each $i \in S$. This is odd, since it means that despite having the power to interfere in the affairs of others, coalition S actually does not fully exercise even its own sovereignty and does not avail payoffs which are *final* and higher for its members. Instead, it acts as if its aim is to facilitate the final payoff desired by player i rather than obtaining higher final payoffs for its members. Given such "benevolent" behavior on the part of coalitions, it is not surprising that there exists a dominance chain from any imputation to any interior imputation and, thus, *at least* every singleton set containing an interior imputation is a stable set.[17]

To conclude, in games with empty cores "almost" every singleton set containing an imputation is a Harsanyi stable set. In other words, despite the fact that each Harsanyi stable set is extremely exclusive, the Harsanyi stable sets as a solution concept lack predictive power because "almost" every imputation is a Harsanyi stable set. This comes directly from the fact that the dominance relation $\gg$ does not require a deviating coalition to aim for an imputation which is at least maximally feasible for it. As seen in the proof of Proposition 4, the deviating coalition S in the last step of each dominance chain chooses an imputation x such that $v(S) < \sum_{i\in S} x$ despite the fact that these payoffs are final and it could have chosen instead an imputation x'such that $v(S) = \sum_{i\in S} x'_i$ and $x'_i > x_i$ for each $i \in S$.

Chwe (1994, p. 310) motivates and introduces another leading and influential farsighted stable solution concept-- called the largest consistent set -- by pointing out that the Harsanyi stable sets are too exclusive. But for games with empty cores it goes to the other extreme in the sense that it includes every interior imputation and, thus, it is "too" inclusive. This follows from the fact that every Harsanyi stable set is a subset of the largest consistent set (Chwe 1994) and Proposition 4 which implies that every singleton set containing an interior imputation is a Harsanyi stable set.[18]

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[17] It may be noted that the proposition does not rule out that a singleton set containing an imputation which is not interior may also be a Harsanyi stable set.

[18] This supplements a result in Bèal et al. (2008, Proposition 1) which shows that the largest consistent set includes every imputation if the game is *strictly superadditive* and has at least *four* players. Proposition 4 here

As in the case of the largest consistent set, Ray and Vohra (2014) motivate and introduce their farsighted stable sets as a modification of the Harsanyi stable sets. Though it does not seem possible to compare the dominance relations underlying the definitions of an IFSS and a farsighted stable set in Ray and Vohra (2014), since the latter is defined over "states" rather than imputations, it might be possible to compare their outcomes. Since every IFSS contains the core and there exist farsighted stable sets which consist of single core imputations, the two may not be disjoint if the core is nonempty, but disjoint if the core is empty. Additional comparisons between the IFSSs and the farsighted stable sets are made in Examples 1-3 below, after we have proved existence and further characterized the IFSSs.

4. Existence and additional characterization of infinitely farsighted stable sets

The purpose of this section is to prove existence and characterize an IFSS for general three-player superadditive TUCF games with empty cores.[19] Though the focus of the paper is on games with *empty* cores, our concepts and analysis might be considered unsatisfactory if existence of IFSSs for games with *nonempty* cores cannot be proved similarly. Accordingly, in this section we also prove existence and characterize IFSSs for general three-player superadditive TUCF games with nonempty cores.

Since, as shown in Section 3, the core is contained in every IFSS and imputations on the boundary of the core are deviations, proving existence of credible deviations in games with empty cores also demonstrates that credible deviations do not necessarily belong to the core. So that is where we begin and prove existence and characterize a set of credible deviations and an IFSS for three-player games with empty cores.

4.1 *Empty cores and credible deviations*

Let (N, v) be a three-player superadditive TUCF game with an empty core. Let the vector (x_1^*, x_2^*, x_3^*) denote a solution to the system of equations

$$x_i + x_j = v(ij),\ \ i, j \in \{1, 2, 3\}. \quad (1)$$

implies that every interior imputation belongs to the largest consistent set if the core is empty and the game has three or more players and not even superadditive.

[19]Three-player games are minimally sufficient for studying coalitional stability. They have been previously studied in Moldovanu (1992), Binmore (1995), and (Maskin (2003) among others.

Then, each player i would be indifferent between forming a two-member coalition with player j or k if division of the worth in coalitions ij and ik are (x_i^*, x_j^*) and (x_i^*, x_k^*), respectively. The vector (x_1^*, x_2^*, x_3^*) has the explicit form

$$x_i^* = \frac{1}{2}[v(ij) + v(ik) - v(jk)], \{i, j, k\} = \{1,2,3\}. \quad (2)$$

Since the unique vector (x_1^*, x_2^*, x_3^*), by definition, is the von Neumann-Morgenstern vector, we shall refer to it as the vNM vector.[20]

Proposition 5 Let (N, v) be a three-player superadditive TUCF game. Then the core of the game is empty if and only if $\frac{1}{2}[v(ij) + v(ik) + v(jk)] > v(123)$.[21]

Proof: Suppose contrary to the assertion that $\frac{1}{2}[v(ij) + v(ik) + v(jk)] > v(123)$ but the core is nonempty. Let (x_1, x_2, x_3) be a core payoff vector. Then, it must satisfy at least the inequalities $x_i + x_j \geq v(ij)$, $i, j \in \{1, 2, 3\}$. But that implies $x_1 + x_2 + x_3 \geq \frac{1}{2}[v(ij) + v(ik) + v(jk)] > v(123)$, which contradicts our supposition that (x_1, x_2, x_3) is a core payoff vector. Next, we show that the core is nonempty if $\frac{1}{2}[v(ij) + v(ik) + v(jk)] \leq v(123)$. If (x_1^*, x_2^*, x_3^*) is such that $x_i^* \geq v(i), i = 1,2,3$, then any feasible payoff vector (x_1, x_2, x_3) such that $x_i \geq x_i^*, i = 1,2,3$, and $x_1 + x_2 + x_3 = v(N)$ is a core payoff vector and thus the core is nonempty. But if $x_i^* < v(i)$ for some i, then $x_j^* \geq v(j)$ and $x_k^* \geq v(k)$ for $j, k \neq i$, since $x_i^* + x_j^* = v(ij) \geq v(i) + v(j)$ and $x_i^* + x_k^* = v(ik) \geq v(i) + v(k)$, by superadditivity. Let $x_i = v(N) - v(jk) \geq v(i) > x_i^*$, by superadditivity. Then, (x_i, x_j^*, x_k^*) is a core payoff vector. This proves that the core is empty only if $\frac{1}{2}[v(ij) + v(ik) + v(jk)] > v(123)$. ■

We shall say that a coalition ij divides its worth according to the vNM vector (x_1^*, x_2^*, x_3^*) if the division of its worth is (x_i^*, x_j^*) and refer to x_i^* as the vNM payoff of player i. Equalities

[20] This vector is also referred to as the (endogenous) outside option vector (see e.g. Harsanyi, 1977) and it may be regarded as a normative solution concept for three-player characteristic function games.

[21] The proof of this proposition is both straight forward and standard, but included here for the sake of completeness.

(1) and (2) imply $x_i^* + v(jk) = \frac{1}{2}[v(ij) + v(ik) + v(jk)] > v(123) \geq v(jk) + v(i)$, since the core is empty and the game is superadditive. Therefore,

$$x_i^* > v(i),\ i \in \{1, 2, 3\}, \tag{3}$$

and, by definition, $x_i^* + x_j^* = v(ij)$. However, the payoff vector (x_1^*, x_2^*, x_3^*) is not feasible, since $x_1^* + x_2^* + x_3^* = \frac{1}{2}[v(ij) + v(ik) + v(jk)] > v(123)$. In fact, it should not be, since otherwise the core will be nonempty, in contradiction to our supposition.

Let Z denote the set of feasible payoff vectors $(x_i^*, x_j^*, \hat{x}_k)$ where

$$\hat{x}_k = v(123) - x_i^* - x_j^* = v(123) - v(jk) \geq v(i), \tag{4}$$

by superadditivity of v, and (z_i, z_j, z_k^*) where

$$z_k^* = v(k) \text{ and } z_i \geq x_i^*,\ z_j \geq x_j^*, \{i, j, k\} = \{1,2,3\}. \tag{5}$$

As seen from (3)-(5), each feasible payoff vector $(x_i^*, x_j^*, \hat{x}_k)$ and (z_i, z_j, z_k^*) is a deviation. Thus,

$$Z = \{(x_i^*, x_j^*, \hat{x}_k), (z_i, z_j, z_k^*), z_i \geq x_i^*, z_j \geq x_j^*, \{i, j, k\} = \{1,2,3\}\}$$

is a set of deviations.

Proposition 6 Let (x_1^*, x_2^*, x_3^*) be the unique vNM vector of a three-player superadditive TUCF game (N, v) with an empty core. Then, $Z = \{(x_i^*, x_j^*, \hat{x}_k), (z_i, z_j, z_k^*), z_i \geq x_i^*, z_j \geq x_j^*, \{i, j, k\} = \{1,2,3\}\}$ is a stable set of credible deviations.

<u>Proof</u>: Note that each imputation $(x_i^*, x_j^*, \hat{x}_k)$ is a deviation, since $x_i^* + x_j^* = v(ij)$. Similarly, each (z_i, z_j, z_k^*) is a deviation, since $z_k^* = v(k)$. We first prove internal stability of Z, i.e., no deviation in Z is dominated by another deviation in Z. Let $x^i \equiv (x_i^*, x_j^*, \hat{x}_k)$ and $z^i \equiv (z_i, z_j, z_k^*)$. Then, for no coalition $S \subset N$ and $x^i, x^j \in Z$, we have $\sum_{h \in S} x_h^i = v(S)$ and $x_h^i > x_h^j, h \in S$. Similarly, for no $x^i, z^j \in Z$, we have a coalition S such that $\sum_{h \in S} x_h^i = v(S)$ and $x_h^i > z_h^j, h \in S$ or $\sum_{h \in S} z_h^j = v(S)$ and $z_h^j > x_h^i, h \in S$. Thus, for no x^i, x^j, we have

$x^i \succ x^j$ and for no $x^i, z^j \in Z$, we have $x^i \succ z^j$ or $z^j \succ x^i$. Furthermore, for no $z^i, z^j \in Z, z^i \succ z^j$. Similarly, no dominance holds in the remaining cases.

We next prove external stability of Z, i.e., if a deviation $y = (y_i, y_j, y_k) \notin Z$, then there is a coalition S and an $x^i \in Z$ such that $\sum_{h \in S} x_h^i = v(S)$ and $x_h^i > y_h, h \in S$. Two cases arise: Since y is a deviation either $y_i + y_j = v(ij)$ for some i and j or $y_k = v(k)$ for some k. In the former case, $y_k = \hat{x}_k < x_k^*$ (see (4) above), since $v(ij) = x_i^* + x_j^*$ and $y_i + y_j + y_k = v(ijk) < x_i^* + x_j^* + x_k^*$. Furthermore, either $y_i < x_i^*$ or $y_j < x_j^*$, since $(y_i, y_j, y_k) \notin Z$. Without loss of generality, let $y_j < x_j^*$. Therefore, $y_k < x_k^*$ and $y_j < x_j^*$ and $x_j^* + x_k^* = v(jk)$. Thus, $(\hat{x}_i, x_j^*, x_k^*) \succ y$ for $(\hat{x}_i, x_j^*, x_k^*) \in Z$. In the latter case, $y_k = z_k^* < x_k^*$ (see (3) and (5) above) and since $(y_i, y_j, y_k) \notin Z$, either $y_i < x_i^*$ or $y_j < y_j^*$. Without loss of generality, let $y_j < y_j^*$. Therefore, $y_k < x_k^*$, $y_j < x_j^*$, and $x_j^* + x_k^* = v(jk)$. Thus, $(\hat{x}_k, x_j^*, x_k^*) \succ y$ for $(\hat{x}_k, x_j^*, x_k^*) \in Z$. ■

It may be noted that, as postulated in footnote 10, the stable set Z contains for each coalition S a deviation y such that $\sum_{i \in S} y_i = v(S)$. It is easily verified that for the motivating example in Section 2, $x_1^* = \$5{,}000, x_2^* = \$10{,}000$, and $x_3^* = \$26{,}000$ are the vNM payoffs. Thus, (5000, 10000, 25000) is a credible deviation, but (4000, 11000, 25000) used in the argument against the stability of the grand coalition is not.

4.1.1 Empty cores and infinitely farsighted stable sets

Given the stable set of credible deviations Z, an IFSS, as Proposition 1 shows, is simply the set that includes Z and all imputations which are not dominated by deviations in Z.

Proposition 7 Let (x_1^*, x_2^*, x_3^*) be the unique vNM vector of a three-player superadditive TUCF game, (N, v), with an empty core. Then the set $Y = \{(x_i, x_j, x_k) \in X : x_i \geq x_i^*, x_j \geq x_j^*, x_k \geq v(k), \{i, j, k\} = \{1,2,3\}\}$ is an IFSS.

Proof: Clearly, the stable set of credible deviations $Z \subset Y$. For each $x = (x_i, x_j, x_k) \in Y$, there exists no credible deviation $y \in Z$ such that $y \succ x$, since $x_i \geq x_i^*, x_j \geq x_j^*$, and $x_k \geq v(k)$. Furthermore, if an imputation $x = (x_i, x_j, x_k) \notin Y$, then there must be at least two players i and j such that $x_i < x_i^*$ and $x_j < x_j^*$. But that implies $(x_i^*, x_j^*, \hat{x}_k) \succ (x_i, x_j, x_k)$ for $(x_i^*, x_j^*, \hat{x}_k) \in Z$. ■

In the motivating example in Section 2, each of the three payoff vectors $(x_1^*, x_2^*, \hat{x}_3) = (5000, 10000, 25000)$, $(x_1^*, \hat{x}_2, x_3^*) = (5000, 9000, 26000)$, and $(\hat{x}_1, x_2^*, x_3^*) = (4000, 10000, 26000)$ belongs to Z and, therefore, immune to credible deviations. In these imputations, two players receive their vNM payoffs and the third player receives the remainder of the worth of the grand coalition. The situation is similar to a game of musical chairs: three players compete for two positions. As a result any two players will rush in to form a coalition and afterwards form the grand coalition with the remaining player and share the resulting additional surplus according to some imputation in the set Y.

Ray and Vohra (2014, Theorem 5) show that farsighted stable sets exist in proper simple games with empty cores and have the structure of discriminatory sets.[22] However, the IFSS for these games may be starkly different as the following example shows.

Example 1 Let (N, v) denote the game $v(S) = 1$ if $|S| \geq 2$ and $v(S) = 0$ otherwise.

This game is superadditive and the core is empty. It is easily seen that $(x_1^*, x_2^*, x_3^*) = (0.5,0.5,0.5)$ is the unique vNM vector and, thus, $Z = \{(0.5, 0.5, 0), (0.5, 0, 0.5), (0, 0.5, 0.5)\}$ is a stable set of credible deviations. Since there is no other imputation which is immune to every credible deviation, the set Z is also an IFSS as well as a vNM stable set, by Corollary to Proposition 2. As Ray and Vohra show, the set of imputations in which one player i always receives a fixed payoff $a_i \in (0,0.5)$ and the other two players share the remaining surplus $1 - a_i$ is a farsighted stable set. Suppose, for the sake of concreteness, that player 1 is the fixed-payoff player. Then, a farsighted stable set consists of imputations x such that $x_1 \in (0, 0.5)$ and either $x_2 < 0.5$ or $x_3 < 0.5$. Thus, $x_i < 0.5$ for at least two players in every farsighted stable set and, therefore, for every imputation x in the farsighted stable set, $y \succ x$ for some $y \in Z$. Since no imputation in a farsighted stable set is dominated by another and the stable set of credible deviations Z is itself an IFSS, it follows that an IFSS is disjoint from a farsighted stable set.

[22] A discriminatory set is a set of the form $D(K, a) = \{x \in \triangle : x_i = a_i \text{ for } i \in K\}$, where $\triangle$ is the n-dimensional unit simplex and $a_i \geq 0$.

4.1.2 *Empty cores and the vNM stable sets*

Proposition 2 shows that every vNM stable set that contains a stable set of deviations is a subset of an IFSS and possibly equal. The following proposition shows that the infinitely farsighted stable set $Y = \{(x_i, x_j, x_k) \in X: x_i \geq x_i^*, x_j \geq x_j^*,\ x_k \geq v(k), \{i,j,k\} = \{1,2,3\}\}$ is not a traditional vNM stable set if the game is strictly superadditive.

Proposition 8 Let (N, v) be a three-player superadditive TUCF game with an empty core. Then the farsighted stable set $Y = \{(x_i, x_j, x_k) \in X: x_i \geq x_i^*, x_j \geq x_j^*,\ x_k \geq v(k), \{i,j,k\} = \{1,2,3\}\}$ is not a vNM stable set if and only if $v(ij) + v(k) < v(ijk)$ for at least one $k \in \{1,2,3\}$.

Proof: We first prove the "if" part. Without loss of generality, let $v(12) + v(3) < v(123)$. Then, $x_1^* + x_2^* + x_3^* > x_1^* + x_2^* + \hat{x}_3 = v(123) > v(12) + v(3)$, since the core is empty. Therefore, $x_2^* + \hat{x}_3 < v(23)$ and $x_3^* > \hat{x}_3 > v(3)$, since $x_1^* + x_2^* = v(12)$, by definition. Now consider $(y_1, y_2, y_3) \equiv (x_1^*, x_2^* + \epsilon, \hat{x}_3 - \epsilon)$ such that $\hat{x}_3 - \varepsilon > v(3)$.
Then, $(y_1, y_2, y_3) \in Y$ and , $x_2^* + \epsilon + \hat{x}_3 - \epsilon = x_2^* + \hat{x}_3 < v(23)$ and there exists a $(z_1, z_2, z_3^*) \in Z$ such that $z_3^* = v(3) < y_3$ and $x_2^* \leq z_2 < x_2^* + \epsilon$, i.e., $(y_1, y_2, y_3) \succ (z_i, z_j, z_k^*)$. This proves that Y does not satisfy internal stability and, therefore, it is not a vNM stable set.[23]

We now prove the "only if" part. Let $v(ij) + v(k) = v(ijk) = x_i^* + x_j^* + \hat{x}_k$ for all i, j, k. Then, since $v(ij) = x_i^* + x_j^*$, it follows that $\hat{x}_k = v(k)$ for $k = 1,2,3$ and the infinitely farsighted stable set Y consists of just three deviations, i.e., $Y = Z = \{(x_i, x_j, x_k): x_i = x_i^*, x_j = x_j^*,\ x_k = v(k), \{i,j,k\} = \{1,2,3\}\}$. Hence, by the Corollary to Proposition 2, Y is a vNM stable set. ■

4.2 *Nonempty cores and credible deviations*

[23] It may be noted that the imputation $(x_1^*, x_2^* + \epsilon, \hat{x}_3 - \epsilon)$ used for the proof is not a credible deviation, since it is not maximally feasible for any coalition.

We next prove existence of an IFSS for general three-player superadditive games with *nonempty* cores by first proving existence of a stable set of credible deviations. We will need the following lemma.

Lemma Let (N, v) be a three-player superadditive TUCF game with a nonempty core and let (x_1^*, x_2^*, x_3^*) be the vNM vector. Then, $c = v(123) - (x_1^* + x_2^* + x_3^*) \geq 0$ and $x_i^* + c \geq v(i)$.

Proof: Since $x_i^* + x_j^* + x_k^* = \frac{1}{2}[v(ij) + v(ik) + v(jk)]$, it follows from Proposition 5 that $c \geq 0$ if the core is nonempty. Clearly, $x_i^* + c = x_i^* + v(ijk) - x_i^* - x_j^* - x_k^* = v(ijk) - v(jk) \geq v(i)$, by supaeradditivity of v. ■

Proposition 9 Let (N, v) be a three-player superadditive TUCF game with a nonempty core. Then, there exists a stable set of deviations which is not unique.

Proof: Since the core is nonempty, $c = v(123) - (x_1^* + x_2^* + x_3^*) \geq 0$, by the Lemma. Let

$$E_1 = \{x \in X: x_1 = x_1^* + c, x_2 < x_2^*, x_3 > x_3^*\};$$
$$E_2 = \{x \in X: x_1 > x_1^*, x_2 = x_2^* + c, x_3 < x_3^*\};$$
$$E_3 = \{x \in X: x_1 < x_1^*, x_2 > x_2^*, x_3 = x_3^* + c\};$$

$$F_1 = \{x \in X: x_1 = x_1^* + c, x_2 > x_2^*, x_3^* - c \leq x_3 < x_3^*\};$$
$$F_2 = \{x \in X: x_1^* - c \leq x_1 < x_1^*, x_2 = x_2^* + c, x_3 > x_3^*\};$$
$$F_3 = \{x \in X: x_1 > x_1^*, x_2^* - c \leq x_2 < x_2^*, x_3 = x_3^* + c\};$$

$$F_1' = \{x \in X: x_1 = x_1^* + c, x_2 > x_2^*, x_3 < x_3^* - c\};$$
$$F_2' = \{x \in X: x_1 < x_1^* - c, x_2 = x_2^* + c, x_3 > x_3^*\};$$
$$F_3' = \{x \in X: x_1 > x_1^*, x_2 < x_2^* - c, x_3 = x_3^* + c\};$$
$$A = \{(v(i), x_j, x_k) \in X: x_j \leq x_j^* + c, x_k \leq x_k^* + c, i = 1,2,3\} \cup \{(x_i^* + c, x_j^*, x_k^*), i = 1,2,3\}$$

Let $Z = \cup_{i=1}^{3} E_i \cup F_i \cup A$. We claim that Z is a stable set of deviations. To that end, let $(x_1^* + c, x_2, x_3) \in E_1$. Since every $(x_1^* + c, x_2, x_3) \in E_1$ is an imputation, $x_1^* + c + x_2 + x_3 = v(123) = x_1^* + c + x_2^* + x_3^*$, i.e., $x_2 + x_3 = x_2^* + x_3^* = v(23)$ and, thus, E_1 is a set of deviations. Similarly, $E_2, E_3, F_i, F_i', i = 1,2,3$, and A are all sets of deviations and, thus, their union Z is a set of deviations.

To prove internal stability of Z, we first show that no deviation in E_1 is dominated by a deviation in Z. No deviation $(x_1^* + c, x_2, x_3)$ in E_1 is dominated by a deviation in E_2 or F_1, since for every deviation $(y_1, x_2^* + c, y_3)$ in E_2 or $(x_1^* + c, z_2, z_3)$ in F_1, $y_3, z_3 < x_3^*$, but $x_3 > x_3^*$. Thus, only a domination through coalition 12 is possible, but that is impossible, since $y_1 \leq x_1^* - c \leq x_1^* + c$, if $y_1 + x_2^* + c \leq v(12)$; no deviation$(x_1^* + c, x_2, x_3)$ in E_1 is dominated by a deviation in E_3 or F_2, since for every deviation $(y_1, y_2, x_3^* + c)$ in E_3 or $(z_1, x_2^* + c, z_3)$ in $F_2, y_1, z_1 < x_1^* + c$, and, thus, only a domination through coalition 23 is possible, but that is impossible, since $x_2^* + x_3^* = v(23)$; no deviation $(x_1^* + c, x_2, x_3)$ in $E_1$is dominated by a deviation in F_3, since for every deviation $(y_1, y_2, x_3^* + c) \in F_3$, $y_1 + y_2 = x_1^* + x_2^*, y_2 \geq x_2^* - c$, and, therefore, $y_1 \leq x_1^* + c$, and no domination is possible through coalition 23 either, since $x_2 + x_3 = x_2^* + x_3^* = v(23)$. Clearly, no deviation $(x_1^* + c, x_2, x_3)$ in E_1 is dominated by another deviation in E_1.

No deviation $(x_1^* + c, x_2, x_3)$ in E_1 is dominated by a deviation $(v(1), y_2, y_3) \in A$, since $v(1) \leq x_1^* + c$ and $x_2 + x_3 = x_2^* + x_3^* = v(23)$; no deviation $(x_1^* + c, x_2, x_3)$ in $E_1$is dominated by a deviation $(y_1, v(2), y_3)$, since $x_2 \geq v(2)$, $y_1 \leq x_1^* + c$, and $x_2 + x_3 = x_2^* + x_3^* = v(23)$; no deviation $(x_1^* + c, x_2, x_3)$ in E_1 is dominated by a deviation $(y_1, y_2, v(3))$, since $x_3 \geq v(3)$, $y_1 \leq x_1^* + c$, and $x_2 + x_3 = x_2^* + x_3^* = v(23)$. Clearly, no deviation $(x_1^* + c, x_2, x_3)$ in E_1 is dominated by a deviation $\left(x_i^* + c, x_j^*, x_k^*\right) \in A$.

Similarly, no deviation in E_2, E_3, F_1, F_2 or F_3 is dominated by a deviation in E_i, $F_i, i = 1,2,3$ or A. To complete the proof for internal stability of Z we also need to show that no deviation in A is dominated by a deviation in E_i or $F_i, i = 1,2,3$ or A.

To that end, no deviation $(v(1), y_2, y_3) \in A$ is dominated by a deviation in E_i or $F_i, i = 1,2,3$ or A, since (i) by definition, $y_3 \leq x_3^* + c$ and, thus, $v(1) + y_2 \geq x_1^* + x_2^* = v(12)$, (ii) similarly, $y_2 \leq x_2^* + c$ and, thus, $v(1) + y_3 \geq x_1^* + x_3^* = v(13)$, and (iii) since $v(1) \leq x_1^* + c$, we have $y_2 + y_3 \geq x_2^* + x_3^* = v(23)$. Similarly, no deviation $(y_1, v(2), y_3)$ or $(y_1, y_2, v(3))$ dominated by a deviation in E_i or $F_i, i = 1,2,3$ or A. No deviation $(v(i), x_j, x_k) \in A$ is dominated by a deviation $(v(j), x_k, x_i)$ or $(v(k), x_i, x_j)$ in A, since (i) $(v(i), x_j, x_k)$ is an imputation, i.e. $x_j \geq v(j)$ and $x_k \geq v(k)$, (ii) $v(i) + x_j \geq v(ij)$, since $x_k \leq x_k^* + c$, (iii) $v(i) + x_k \geq v(ik)$, since $x_j \leq x_k^* + c$, and (iv) $x_j + x_k \geq v(jk)$, since

$v(i) \le x_i^* + c$, by the Lemma. Clearly, no deviation $(x_i^* + c, x_j^*, x_k^*) \in A$ is dominated by a deviation in E_i $F_i, i = 1,2,3$ or A.

To prove external stability of Z, note that Z includes all possible deviations in the game except those in F_1', F_2', and F_3'. We first show that each deviation in F_3' is dominated by a deviation in E_1. Given a deviation $(x_1, x_2, x_3^* + c)$ in F_3', let y_2 be such that $x_2 < y_2 < x_2^* - c, y_3 > x_3^* + c$, and $y_2 + y_3 = x_2^* + x_3^*$. Then, $(x_1^* + c, y_2, y_3)$ is a deviation in E_1 which dominates $(x_1, x_2, x_3^* + c)$. Similarly, deviations in F_1' (resp. F_2') are dominated by deviations in E_2 (resp. E_3).

It remains to be shown that deviations by single-player coalitions not in the set Z are also dominated by deviations in Z. A deviation $(v(1), x_2, x_3)$ by the single-player coalition $\{1\}$ does not belong to the set Z only if either $x_3 < x_3^* + c$ and $x_2 > x_2^* + c$ or $x_3 > x_3^* + c$ and $x_2 \ge (<) x_2^* + c$.[24] Since $v(1) + x_2 + x_3 = x_1^* + x_2^* + x_3^* + c$, either $v(1) + x_3 < x_1^* + x_3^*$ and/or $v(1) + x_2 < x_1^* + x_2^*$. Thus, $(v(1), x_2, x_3)$ is dominated by a deviation in either E_2 or F_2 and /or E_3 or F_3. Similarly, deviations $(x_2, v(2), x_3)$ and $(x_1, x_2, v(3))$ by single player coalitions $\{2\}$ and $\{3\}$, respectively, are also dominated by deviations in Z.

Let $Z' = \cup_{i=1}^{3} G_i \cup H_i \cup A$, where

$$G_1 = \{x \in X: x_1 = x_1^* + c, x_2 > x_2^*, x_3 < x_3^*\};$$
$$G_2 = \{x \in X: x_1 < x_1^*, x_2 = x_2^* + c, x_3 > x_3^*\};$$
$$G_3 = \{x \in X: x_1 > x_1^*, x_2 < x_2^*, x_3 = x_3^* + c\};$$

$$H_1 = \{x \in X: x_1 = x_1^* + c, x_2^* - c \le x_2 < x_2^*, x_3 > x_3^*\};$$
$$H_2 = \{x \in X: x_1 > x_1^*, x_2 = x_2^* + c, x_3^* > x_3 \ge x_3^* - c\};$$
$$H_3 = \{x \in X: x_1^* - c \le x_1 < x_1^*, x_2^* > x_2, x_3 = x_3^* + c\};$$

$$H_1' = \{x \in X: x_1 = x_1^* + c, x_2 < x_2^* - c, x_3 > x_3^*\};$$
$$H_2' = \{x \in X: x_1 > x_1^*, x_2 = x_2^* + c, x_3 < x_3^* - c\};$$
$$H_3' = \{x \in X: x_1 < x_1^* - c, x_2 > x_2^*, x_3 = x_3^* + c\};$$
$$A = \{(v(i), x_j, x_k) \in X: x_j \le x_j^* + c, x_k \le x_k^* + c, i = 1,2,3\} \cup \{(x_i^* + c, x_j^*, x_k^*), i = 1,2,3\}$$

[24] If $x_3 = x_3^* + c$, then $(v(1), x_2, x_3)$ belongs to either E_3 or F_3 and, thus, Z, since $v(1) \le x_1^* + c$, by the Lemma.

Then, Z' is a stable set of credible deviations. The proof for this is analogous to the proof for the stability of Z and hence not included. ■

4.2.1 *Nonempty cores and infinitely farsighted stable sets*

Let Y (resp. Y') denote the set of all imputations of a three-player superadiitive TUCF game (N, v) with a nonempty core which are not dominated by the deviations in the stable set of deviations Z $(\text{resp.} Z')$ as defined in Proposition 9. Then, by Proposition 1, Y and $Y'$are IFSSs of (N, v). Furthermore, since $Z \neq Z'$, $Z \subset Y$, and $Z' \subset Y'$, neither Y is a subset of Y' nor $Y'$is a subset of Y.

At this point, it seems worth illustrating and contrasting the IFSSs, the vNM stable sets, and the farsighted stable sets by an example.[25] In this example, $Y = Z$ and $Y' = Z'$ and, thus, the two IFSSs are vNM stable sets, but there is also a vNM stable set which is not an IFSS.

Example 2 Let (N, v) denote the three-player game: $v(i) = 0, v(23) = 0, v(12) = v(13) = v(N) = 1$.

It is easily seen that the core of this game is nonempty and consists of the unique vNM vector $(x_1^*, x_2^*, x_3^*) = (1,0,0)$ which has player 1 receiving the entire surplus of the game. Since $x_1^* + x_2^* + x_3^* = v(N)$, i.e., $c = 0$ and $x_2^* = x_3^* = v(2) = v(3) = 0$, the sets E_1, E_2, F_1, F_2, F_3, F_1', and F_3' are all empty. Only the sets E_3 and F_2' are nonempty. Accordingly, the stable set of deviations $Z = \{(\alpha, 1-\alpha, 0): 0 \leq \alpha \leq 1\}$. Similarly, $Z' = \{(\alpha, 0, 1-\alpha)\}$, since all but the sets G_2 and H_3' are empty. Both Z and Z' are IFSSs, since there are no imputations outside $Z(\text{resp. } Z')$ which are not dominated by a deviation in Z $(\text{resp.} Z')$. Player 1 (the veto player) receives a varying payoff in either of these two IFSSs. Since player 1 cannot get a positive payoff without the help of player 2 or 3, it makes sense that the entire surplus of the game does not accrue to player 1. However, competition between players 2 and 3 can drive their shares of the surplus to zero and the two IFSSs indeed include that possibility as the imputation $(1,0,0)$ belongs to both Z and Z'.

In contrast, as Ray and Vohra (2014) show, every imputation close to the unique core imputation $(1,0,0)$ is a farsighted stable set, but no farsighted stable set actually includes

[25] The example has previously appeared in Owen (1982, p.180), Lucas (1992), and Ray and Vohra (2014).

$(1,0,0)$ and exhibits a varying payoff for the veto player. While in an IFSS either player 2 or 3 is the fixed payoff player, in a farsighted stable set player 1 is the fixed payoff player and both players 2 and 3 receive varying payoffs.

By the Corollary to Proposition 2, the two IFSSs Z and Z' are (discriminatory) vNM stable sets. However, the set $V = \{(1-t, t\alpha, t(1-\alpha)) : 0 \leq t \leq 1\}$ for each fixed $0 \leq \alpha \leq 1$, is also a vNM stable set (see Owen, 1982), but not an IFSS except when $\alpha = 0$ or $\alpha = 1$. That is because if $\alpha \neq 0{,}1$, then V contains no stable set of credible deviations and, in fact, no deviation at all (except the lone core deviation $(1,0,0)$). A final important difference is that V includes the imputation $(0, \alpha, 1-\alpha)$ in which the two non-veto players extract the entire surplus of the game, despite the fact that their coalition can achieve nothing without player 1 (since $v(23) = 0$) and they have to compete with each other to form a coalition with player 1.

4.2.2 *Convex games and infinitely farsighted stable sets*

Shapley (1971) shows that every convex game has a nonempty core and the core is the unique vNM stable set. Example 2 above shows that not every vNM stable set is an IFSS. But it is, if the game is convex, as the following proposition shows.

Proposition 10 Let (N, v) be a convex TUCF game. Then the imputations on the boundary of the core form a stable set of deviations and the core is the unique infinitely farsighted stable set.

Proof: Since a deviation belongs to the core if and only if it belongs to the boundary of the core, we prove that the deviations belonging to the core form a stable set. This set is nonempty, since the core, by definition, is a closed set. The set satisfies internal stability, since no core imputation, by definition, is dominated by a core deviation. To prove external stability, let x be a deviation which does not belong to the core. Then, by Theorem 8 in Shapley (1971), there exists a core imputation y and a coalition S such that $\sum_{i \in S} y_i = v(S)$ and $y_i > x_i$ for each $i \in S$, i.e., there is a deviation y in the set which dominates deviation x. Thus, the imputations on the boundary of the core form a stable set of deviations.

Next, since every imputation outside the core is dominated by a core deviation and no core imputation, by definition, is dominated by a core deviation, the core imputations, by Proposition 1, form an IFSS. Finally, since every IFSS contains the core and no IFSS, by Proposition 2, is a proper subset of another, it follows that the core is the unique IFSS. ■

Propositions 2 and 10 together lead to a reinterpretation of Shapley's (1971) result that if the game is convex, then the core coincides with the unique vNM stable set. Specifically, since every vNM stable set in general contains the core, every vNM stable set of a convex game, by Proposition 10, contains a stable set of deviations and, thus, by Proposition 2, contained in an IFSS, which, by Proposition 10, is unique and coincides with the core. Hence, if the game is convex, every vNM stable set is equal to the core.

Proposition 10 shows that convexity of a game is a sufficient condition for the existence and uniqueness of an IFSS. It is also sufficient for the existence of farsighted stable sets, since the core of a convex game has a nonempty interior and as Ray and Vohra (2014) show every imputation in the interior of the core is a farsighted stable set. However, as the following example illustrates, the farsighted stable sets and the IFSS may still differ, though not as sharply as they do in Examples 1 and 2 when the core is either empty or has no interior.[26]

Example 3 Let (N, v) denote the three-player game: $v(S) = 3$ for every two-player coalition S and $v(N) = 6$.

Clearly, the game is convex and, therefore, the core is the unique IFSS and, by Proposition 10, the imputations on the boundary of the core form a stable set of credible deviations.[27] In contrast, as Ray and Vohra (2014) show, each imputation in the interior of the core and only an imputation in the interior of the core is a farsighted stable set. Thus, the union of all farsighted stable sets is a proper subset of the unique IFSS and excludes all imputations on the boundary of the core which form exactly the stable set of credible deviations contained in the IFSS.

[26] The example has previously appeared in Ray and Vohra (2014).
[27] Since the game has three players, this can also be seen directly from Proposition 9. Since, as can be easily checked, convexity of the game implies $x_i - c \geq 0$ and, therefore, the sets F_i' or H_i', $i = 1,2,3$ are all empty and the stable set Z or Z' is equal to the set of all core deviations.

5. Conclusion

In this paper, we introduced and studied a concept of a stable set which has powerful farsighted stability properties and accordingly named it an infinitely farsighted stable set (IFSS). Though motivated and defined differently, the IFSSs and vNM stable sets are closely related in that every vNM stable set which contains a stable set of credible deviations is contained in an IFSS and in some cases, as examples 1-3 show, actually an IFSS. This means that Harsanyi's (1974) critique of vNM stable sets is not entirely valid, since at least a subset of the vNM stable sets also have powerful farsighted stability properties and we know the exact distinguishing feature of such vNM stable sets. Though the Harsanyi stable sets may exist even in games with empty cores, they do not have much predictive power – quite apart from their criticism already made in Ray and Vohra (2014). Similarly, the largest consistent set (Chwe, 1994) – another farsighted stability concept – includes at least *all interior* imputations if the core is empty and the game has three or more players and not necessarily strictly superadditive.

The general concept of credible deviations also motivated and introduced in this paper is of an independent interest and can be extended to define an analogous solution concept for strategic games (see Chander, 2015). The existence of a stable set of deviations is the key to the definition and existence of an IFSS. A set is infinitely farsighted stable if it can be decomposed into a stable set of deviations and a -- possibly empty -- set of imputations which are not dominated by any deviation in the stable set and, thus, are not deviations themselves. For this reason, a vNM stable set may differ from an IFSS, since, as Example 2 shows, the former may not include a stable set of credible deviations. Similarly, an IFSS may differ from the union of farsighted stable sets (Ray and Vohra, 2014), since the latter, as examples 1-3 show, may also not include a stable set of credible deviations.

Except for a result in Ray and Vohra (2014), there is not much else in the previous literature on the outcomes of games with *empty* cores. Maskin (2003, p.3) provides an example of a three-player characteristic function game to assert that emptiness of the core is a necessary but not a sufficient condition for instability of the grand coalition. In this paper, we proved existence of IFSSs for general three-player superadditive transferable utility characteristic function games with *empty* cores and, thus, shown that emptiness of the core, in fact, is not a sufficient condition for instability of the grand coalition in any *general* three-player game. In addition, we showed that the core is the unique IFSS in every convex game with any number of players.

These positive existence and characterization results bring to the fore an important question regarding games with more than three players and *empty* cores: Do IFSSs generally exist for games with more than three-players and empty cores?[28] It seems that strict superadditivity of the game (thus the grand coalition is uniquely efficient) with an empty core might be a sufficient condition for the existence of an IFSS.[29] But a serious exploration of this question must be left open as a future research project. At stake is the Coase theorem.

[28] It is worth noting that the game in Lucas (1968) does not have an empty core.
[29] Since convex games have nonempty cores, convexity cannot be a sufficient condition,

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