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ORIGINAL PAPER

Stability of the merger-to-monopoly and a core concept for partition function games

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Abstract
This paper is concerned with an old question: Will oligopolistic firms have incentives to merge to monopoly and will the monopoly, if the firms indeed merge, be stable? To answer this question, I motivate and introduce a new core concept for a general partition function game and prove stability of the merger-to-monopoly by applying the new core concept, labelled the strong-core, to Cournot oligopoly modelled as a partition function game. The paper shows that the Cournot oligopoly with any finite number of homogeneous firms without capacity constraints admits a non-empty strong-core and so does the Cournot oligopoly of not necessarily homogeneous firms with capacity constraints that are equal to their "historical" outputs. These results imply that oligopolistic firms will have incentives to merge to monopoly both in the long- and short-run and if the firms indeed merge to monopoly, the merger-to-monopoly will be stable.

Keywords Oligopoly · Cartel · Monopoly · Partition function game · Core

JEL Classification C72 · D43 · L12–13

1 Introduction

Numerous studies have focused on conditions under which horizontal mergers among firms in a Cournot oligopoly can be profitable for participating firms. Notable contributions include Salant et al. (1983), Perry and Porter (1985), Deneckere

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and Davidson (1985) and Farrell and Shapiro (1990), among others. Most of these studies focus on incentives of oligopolistic firms to merge to monopoly. In particular, Deneckere and Davidson conclude: "… short of antitrust policy, the industry would concentrate almost completely towards monopoly." However, the question of stability of mergers, especially of the merger-to-monopoly, has been somewhat ignored.

In this paper, I argue that mere profitability of a merger does not guarantee that the merger will be stable. Although oligopolistic firms do have incentives to merge into a single multi plant firm because of higher monopoly profits, but still some firm or coalition of firms may decide not to merge to monopoly as it may conclude that its greatest profit potential lies in remaining separate. Thus, the merger-to-monopoly, although profitable, can be stable only if the monopoly profits can be split in a way that no firm or coalition of firms will have incentive to leave the merger and become an independent market player. As will be shown, the existence and possibility of such a split of monopoly profits depends on whether or not the core of the Cournot oligopoly modelled as a partition function game (Thrall and Lucas 1963) is nonempty.[1] Indeed, Maskin (2003, p. 3) asserts that if there are games in which the grand coalition (i.e. the merger-to-monopoly) is not stable it is *only* among coreless games that we will find them. In other words, a nonempty core is a *sufficient* condition for stability of the grand coalition and an empty core is a *necessary* condition for its instability.

Thus, if the Cournot oligopoly modelled as a partition function game admits a nonempty core, then we need to look no further as that implies that the merger-to-monopoly is stable. But the answer is a lot more complicated than that, because for *partition function games* (also known as games with "externalities"), there exist many alternative core concepts and the core may or may not be empty depending on which core concept is used.[2] The reason for many alternative core concepts is that in partition function games, unlike coalitional games, a deviating coalition (i.e. a group of players that decide to act together as one unit relative to the rest of players) must take into account the coalition(s) that may be formed by the players in the complement subsequent to its deviation, as the payoff/worth of a coalition depends on the entire partition. For this reason, different assumptions regarding the partition that may be formed by the players subsequent to a deviation lead to different core concepts. One simple definition of the core of a partition function game, known as the γ-core, is given by assuming that all players/firms outside the deviating coalition form singletons (Chander and Tulkens 1997). Another equally simple definition, known as the δ-core, is given by assuming that the players/firms outside the deviating coalition form one single coalition of their own (Maskin 2003).

[1] See Kóczy (2018: Ch. 11) for an excellent review of the literature on applications of partition function games to Cournot oligopoly.

[2] While for coalitional games there exists an adequately and widely accepted core concept, this is not true for partition function games.

Rajan (1989) seems to have been the first to apply cooperative game theory and the core to Cournot oligopoly.[3] For reasons of tractability, Rajan (1989) restricts the application to a symmetric oligopoly with at most 3 or 4 firms and concludes that the incentive for all firms to merge depends on the core concept. Rajan (1989) considers three core concepts: the α-, γ-, and δ-cores (without naming the latter two), and justifies their assumptions regarding the partition that the firms/players may form subsequent to a deviation as market's "ethos", which cannot be explained by economic theory (Jaskow 1975; Shubik 1975). Rajan (1989) shows that if the number of firms is three, then the δ-core of the Cournot oligopoly is empty, but the γ-core is not, highlighting thereby the significance of the role played by market's ethos in the stability analysis of the merger-to-monopoly.

This paper first introduces a new core concept for a general partition function game, labelled the strong-core, which is independent of market's ethos in that the strong-core does not make any ad hoc assumption regarding the partition that may be formed by the players subsequent to a deviation; instead, it assumes that any partition may be formed, except that the partition must include the deviating coalition. More specifically, the strong-core is the set of all payoff vectors that are feasible for the grand coalition and such that in *any* possible partition that contains the deviating coalition either the deviating coalition itself or some other coalition with at least two players is worse-off.

I show that the symmetric Cournot oligopoly with *any number* of firms without capacity constraints admits a non-empty strong-core and so does the Cournot oligopoly with any number of not necessarily homogeneous firms with capacity constraints equal to their "historical" outputs. These results imply that both in the long run when production capacities can be expanded and in the short run when they cannot be, oligopolistic firms, independently of market's ethos, will have incentives to merge to monopoly and if the firms indeed merge to monopoly, the merger-to-monopoly will be stable.

The contents of this paper are as follows: Sect. 2 introduces the strong-core and compares it with the existing core concepts. It shows that a well-known class of symmetric games admit non-empty strong-cores. Section 3 proves that the strong-core of the Cournot oligopoly with any finite number of homogeneous firms is non-empty and so is the strong-core of the Cournot oligopoly with any finite number of not necessarily homogeneous firms, if each firm faces a capacity constraint equal to its historical output. Section 4 draws the conclusion.

[3] See Corchòn and Marini (2018) for more recent applications of cooperative game theory and the core to oligopolistic markets.

2 Core concepts for partition function games

The purpose of this section is to first review the existing core concepts for partition function games and then motivate and introduce the strong-core. The next section applies the strong-core to Cournot oligopoly by converting the oligopoly into a partition function game.

Let $N = \{1, \dots, n\}, n \geq 3$, denote the set of players. A set $P = \{S_1, \dots, S_m\}$ is a partition of N if $S_i \cap S_j = \varnothing$ for all $i, j = 1, \dots, m, i \neq j$, and $\cup_{i=1}^{m} S_i = N$. I shall denote the finest partitions of N, S, and $N \backslash S$ by $[N]$, $[S]$, and $[N \backslash S]$, respectively, the cardinality of set S by $|S|$, and (to save on notation) singleton sets $\{S\}$ and $\{[S]\}$ simply by S and $[S]$, respectively, whenever no confusion is possible.

A partition function game is a pair (N, v) where $v(S;P)$ is the worth of coalition S, if P is the partition and S is a member of P. As the worth of a coalition in a partition function game depends on the partition to which it belongs, the partition function games are sometimes referred to as games with "externalities". A partition function game in which the worth of each coalition is independent of the partition and depends *only* on the coalition is a special case and can be adequately represented by a coalitional game.

Given a partition function game (N, v), a payoff vector $x = (x_1, \dots, x_n)$ is feasible for the grand coalition if $\sum_{i \in N} x_i = v(N;N)$. I shall denote a payoff vector that is feasible for the grand coalition by x.

2.1 The existing core concepts

The core, proposed by Gillies (1953) is a leading and influential solution concept for coalitional games. But in partition function games, unlike coalitional games, a deviating coalition has to take into account what other coalitions may form in the complement subsequent to its deviation, since its worth/payoff depends on the entire partition. For this reason, all existing core concepts for a partition function game make one or another ad hoc assumption regarding the coalition(s) that may be formed by the players outside a deviating coalition—leading to alternative core concepts depending on the assumption made in this regard. In this subsection, I first review the two most widely used core concepts for partition function games.

Definition 1 The γ-core of a partition function game (N, v) is the set of feasible payoff vectors x such that for every deviating coalition S and partition $\{S, [N \backslash S]\}$, $\sum_{i \in S} x_i \geq v(S;\{S, [N \backslash S]\})$. That is, each γ-core payoff vector is such that every deviating coalition S is worse-off in the partition $\{S, [N \backslash S]\}$.

The γ-core (Chander and Tulkens 1997; Chander 2018a, b), motivated and introduced as an improvement over the traditional α- and β-cores, assumes formation of a specific partition subsequent to a deviation from the grand coalition in that if coalition S deviates from the grand coalition then the players form the partition $\{S, [N \backslash S]\}$.

Definition 2 The δ-core of a partition function game (N, v) is the set of feasible payoff vectors x such that for every deviating coalition S and binary partition $\{S, N\backslash S\}$, $\sum_{i\in S} x_i \geq v(S;\{S, N\backslash S\})$. That is, each δ-core payoff vector is such that every deviating coalition S is worse-off in the binary partition $\{S, N\backslash S\}$.

The δ-core (Maskin 2003), like the γ-core, also assumes formation of a specific partition, albeit a different one, subsequent to a deviation from the grand coalition. Specifically, if coalition S deviates from the grand coalition then the players form the binary partition $\{S, N\backslash S\}$.

It is worth noting that the δ-core actually requires not only a deviating coalition S but also the complementary coalition $N\backslash S$ to be worse-off in the binary partition $\{S, N\backslash S\}$. That is because if the deviating coalition is $N\backslash S$, then, by definition of the δ-core, coalition $N\backslash S$ should be worse-off in the binary partition $\{N\backslash S, S\} = \{S, N\backslash S\}$.

It may be mentioned that the γ- and δ-cores, as defined above, are not the same as similarly named concepts in Hart and Kurz (1983). In contrast to the approach in the present paper, Hart and Kurz (1983) do not require the sum of payoffs of the members of a coalition in a partition to be equal to the worth of the coalition in the partition. This comes about from their "efficiency" axiom according to which the worth of the grand coalition is assumed to accrue as the total payoff of all players in every partition.

Apart from the δ-core, the traditional α- and β-cores (Aumann 1961) not only assume formation of the binary partition $\{S, N\backslash S\}$ subsequent to a deviation by coalition S, but also assume predatory behaviour on the part of the complementary coalition $N\backslash S$.[4] There are also core concepts in which the partition formed by the players subsequent to a deviation by a coalition is determined endogenously, but by imposing an exogenous selection criterion. These include the c-core in which the players in the complement of a deviating coalition are assumed to form coalitions such that the payoff of the deviating coalition is minimized and the r-core in which the players in the complement are assumed to form coalitions such that the sum of their own payoffs is maximized.[5] Similarly, Huang and Sjöstörm (2003) and Kóczy (2007) introduce a core concept, also named the r-core or the recursive core, by (exogenously) imposing a consistency requirement on the partition that may be formed by the players.

In sum, all existing core concepts for partition function games make one or another ad hoc assumption concerning the partition that may be formed by the players subsequent to a deviation by a coalition. For this reason, none of the existing core concepts, although used in many applications, has yet become an adequately and widely accepted core concept for partition function games.

[4] See Chander (2007, 2018a) and Ray and Vohra (1997) for additional criticisms of the α- and β-cores.

[5] See Hafalir (2007) for formal definitions of both c- and r-cores.

2.2 A new core concept

I propose a new core concept which, unlike the existing core concepts, does not rule out formation of any partition subsequent to a deviation by a coalition, except that the partition must include the deviating coalition. In other words, the new core concept is independent of the partition that may be formed by the players subsequent to a deviation by a coalition.

Definition 3 The strong-core of a partition function game (N, v) is the set of feasible payoff vectors x such that for every deviating coalition S and any partition $P \ni S$ either $v(S;P) \leq \sum_{i \in S} x_i$ or $v(T;P) \leq \sum_{i \in T} x_i$ for some other coalition $T \in P$ with at least two players. That is, every strong-core payoff vector is such that either the deviating coalition S or some other coalition T with at least two players is worse-off in any partition $P \ni S$ that could possibly be formed by the players subsequent to the deviation by S.

The definition of the strong-core is mathematically comparable to the definitions of the γ- and δ-cores in that the strong-core, unlike the γ- and δ-cores, does not assume formation of any specific partition subsequent to a deviation by a coalition—any partition may form; no partition is ruled out, except that the partition must contain the deviating coalition. Thus, if S is the deviating coalition, the resulting partition can be $\{S, [N \backslash S]\}$ or $\{S, N \backslash S\}$ or any other partition $P \ni S$.

As the concept is new, I further interpret and justify the strong-core. One interpretation of the strong-core is as follows: Suppose that a feasible payoff vector x is under discussion of the players and, for the reason made clear below, it must be *collectively* accepted or rejected. Now suppose that a coalition S thinks that it can do better than x provided the players form a particular partition $P \ni S$. But because in partition function games, unlike coalitional games, the worth/payoff of each coalition in partition P depends on the entire partition (i.e. the coalition structure), coalition S must convince all other players to form partition P; and for this reason the decision to accept or reject the feasible payoff vector x has to be taken *collectively* by all players. The question is: what are the *minimum* conditions that partition $P \ni S$ must fulfil for S to succeed in convincing all other players to form partition P. A *necessary* condition for all other players to agree to form partition $P \ni S$ is that no coalition with two or more players in partition P should be worse-off compared to the feasible payoff vector x. Every strong-core payoff vector x, by definition, is such that there is no partition $P \ni S$ that satisfies even this necessary condition. Thus, every strong-core payoff vector is "strongly" stable.

Another interpretation of the strong-core is as the set of feasible payoff vectors x that cannot be "blocked" by any partition $P \ni S$ in which at most singleton coalitions are worse-off.

The former interpretation of the strong-core explains why a deviating coalition may care that no coalition with two or more players should be worse-off in a partition. But why should a deviating coalition not care similarly that no *singleton* coalition either should be worse-off in a partition? The reason is that a singleton coalition, unlike a coalition with two or more players, cannot break-up into two or more coalitions and upset a partition in which it is worse-off. Furthermore, a singleton coalition may not be able to upset a partition in which it is worse-off by merging with another coalition either because no coalition in the partition may be willing to merge with it. In contrast, a coalition with two or more players can *always* upset a partition in which it is worse-off simply by breaking up into two or more coalitions.[6] To illustrate, consider a 3-player partition function game and a partition P consisting of a pair and a singleton, say $P = \{\{1,2\}, \{3\}\}$. Suppose singleton coalition $\{3\}$ is worse-off in partition P compared to a feasible payoff vector x and coalition $\{1,2\}$ is better-off in partition P. Then, singleton coalition $\{3\}$ cannot prevent blocking of x by partition $\{\{1,2\}, \{3\}\}$, although it will be worse-off in partition $\{\{1,2\}, \{3\}\}$, because it can neither break apart nor merge with coalition $\{1, 2\}$; coalition $\{1, 2\}$will not be willing to merge with $\{3\}$ as coalition $\{1, 2\}$ will be worse-off in the resulting partition. Thus, partition $P = \{\{1,2\}, \{3\}\}$ may be used by coalition $\{1, 2\}$ to block x without the consent of the singleton coalition $\{3\}$, as $\{3\}$ cannot upset the partition.

The illustrative example shows that restricting blocking to partitions in which no singleton coalition either is worse-off may curb, at least in some cases, the inherent "power" of coalitions to block, and thereby lead to a core concept that is too weak. To see this consider an alternative definition of the core: the core is the set of feasible payoff vectors x such that in every partition that may possibly be formed by the players subsequent to a deviation either the deviating coalition or some other coalition (singleton or non-singleton) is worse-off. This so-defined core is a "too weak" concept as it has no bite in at least the common class of games in which the grand coalition is the unique efficient coalition structure, i.e. $v(N;N) > \sum_{i=1}^{m} v\left(S_i;P\right)$ for every partition $P = \left\{S_1, \dots, S_m\right\} \neq \{N\}$. For this common class of games, which include the Cournot oligopoly modelled as a partition function game, the so-defined core consists of *all* feasible payoff vectors.

Finally, as is easily seen, the strong-core of a partition function game reduces to the core of a coalitional game if the worth of every coalition is independent of the partition to which it belongs and the partition function is adequately represented by a coalitional function. This means the strong-core is an extension of the standard core of a coalitional game to the more general partition function games, as the partition function games include the coalitional games as a special case.

2.3 A consistency property

The γ-core, proposed originally as an improvement over the classical α- and β-cores, has been applied to a wide range of economic models including oligopoly. See e.g. Kóczy (2018), Helm (2001), Lardon (2012) and Stamatopoulos

[6] This argument is reminiscent of the premise in Salant et al. (1983) that a market structure in which some cartel (i.e. a coalition with two or more firms) is worse-off cannot be an equilibrium market structure.

(2016) among others. Thus, a pertinent question is whether the strong-core is consistent with the γ-core.

Proposition 1 *Let (N, v) be a partition function game. Then, the strong-core is generally a subset of the γ-core, but is not generally equal to the γ-core.*

Proof Let x be a strong-core payoff vector. Since in every partition $\{S, [N \backslash S]\} \neq [N]$ at most coalition S is a non-singleton, it follows from the definition of the strong-core payoff vectors that $\sum_{i \in S} x_i \geq v(S; \{S, [N \backslash S]\}$ for every non-singleton coalition S. Furthermore, this inequality also holds for every singleton coalition S, since if the deviating coalition S is a singleton then the partition $\{S, [N \backslash S]\}$ contains no non-singleton coalition and, therefore, by definition of the strong-core, the deviating coalition S itself must be worse-off. Thus, x is also a γ-core payoff vector. But every γ-core payoff vector is not a strong-core payoff vector, as the following example shows.

Let $N = \{1,2,\ldots,5\}, v(N;N) = 13$, $v(S;\{S, [N \backslash S]\}) = 2.4|S|$, $v(S;\{S;N \backslash S\}) = 2.6|S|$ for $|S| < 4$, $v(S;\{S;N \backslash S\}) = 2.4|S|$ for $|S| = 4$, for each partition $P = \{\{ij\}, \{kl\}, \{m\}\}$, $v(\{ij\};P) = v(\{kl\};P\} = 6$ and $v(\{m\};P) = 1$, for each partition $P = \{\{i\}, \{j\}, \{k\}, \{lm\}\}$, $v(\{i\};P) = 1$, and for each partition $P = \{\{i\}, \{j\}, \{klm\}\}, v(\{i\};P) = 1$.

In this game, the feasible payoff vector $(x_1, x_2, \ldots, x_5) = (2.6, 2.6, \ldots, 2.6)$ belongs to the γ-core and thus the γ-core is nonempty. But the strong-core is empty. This is seen as follows: A feasible payoff vector $(x_1, x_2, \ldots, x_5)$, by definition, belongs to the strong-core only if $\sum_{i \in N} x_i = 13$, $x_i \geq 2.4$, $i = 1,2,\ldots,5$, and at least for the partition $P = \{\{12\}, \{34\}, \{5\}\}$, either $v(\{12\};P) \leq x_1 + x_2$ or $v(\{34\};P) \leq x_3 + x_4$. But there exists no such feasible vector, because $x_i \geq 2.4$, $i = 1,2,\ldots,5$ and, therefore, $x_1 + x_2 = 13 - x_3 - x_4 - x_5 \leq 5.8 < v(\{12\};P)$ and $x_3 + x_4 = 13 - x_1 - x_2 - x_5 \leq 5.8 < v(\{34\};P)$. Hence, the strong-core is empty, but the γ-core is not. This proves that the strong-core is a strict subset of the γ-core. ■

Intuitively, the strong-core is generally a subset of the γ-core because, unlike the γ-core, the strong-core, by definition, does not rule out formation of any partition subsequent to a deviation. For instance, in the proof of Proposition 1, the strong-core, unlike the γ-core, has to take into account every coalitional worth $v(S;P)$ in partitions $P = \{\{ij\}, \{kl\}, \{m\}\}, S \in P$, $i,j,k,l,m \in \{1,2,3,4,5\}$. Thus, externalities play a greater role in the determination of the strong-core than in the determination of the γ-core.

Proposition 1 can be viewed as an extension of a consistency property of the existing core concepts in that it is known (Chander 2018a) that for a general partition function game, γ-core $\subset$ β-core $\subset$ α-core. Proposition 1 implies that for a general partition function game, strong-core $\subset$ γ-core $\subset$ β-core $\subset$ α-core. This is a nice consistency property for the strong-core to have, because it implies that applications of the strong-core are not inconsistent with applications of the familiar α-, β-, and γ-cores. As will be shown, the strong-core of the Cournot oligopoly modelled as a partition function game is a strict subset of the γ-core and is thus also a strict subset of the α- and β-cores.

Similarly, on the one hand, the strong-core is a stronger concept than the δ-core, because the δ-core, by definition, rules out formation of all but binary partitions subsequent to a deviation. But, on the other hand, the strong-core is a weaker concept, because, as noted in the second paragraph following Definition 2, the strong-core, unlike the δ-core, does not require both coalitions in every binary partition to be worse-off. For this reason, the strong-core and the δ-core are not *generally* comparable except in some special cases of interest, as shown below.

Yi (1997) and Maskin (2003) note that in most applications the partition function games can be classified into two classes, namely, as games with "positive" or "negative" externalities.

A partition function game (N, v) has *positive* (*negative*) externalities if for every partition $P = \{S_1, \dots, S_m\}$ and $S_i, S_j \in P$, we have $v(S_k;P\backslash\{S_i, S_j\} \cup \{S_i \cup S_j\}) \geq (\leq) v(S_k;P)$ for each $S_k \in P, k \neq i, j$.

In words, a partition function game has positive (negative) externalities if a merger between any two coalitions in a partition increases (decreases) the worth of every other coalition in the partition.[7] It is known that for games with positive externalities, δ-core $\subset$ γ-core. I now establish an additional inclusion relationship.

Proposition 2 (a) *For partition function games with positive externalities, δ-core $\subset$ strong-core $\subset$ γ-core, and* (*b*) *for partition function games with negative externalities, strong-core $=$ γ-core $\subset$ δ-core.*

Proof (a) First, suppose contrary to the assertion that in a game with positive externalities a δ-core payoff vector $(x_1, \dots, x_n)$ does not belong to the strong-core. Then, because $(x_1, \dots, x_n)$ belongs to the δ-core, for every coalition $S \subset N$ and binary partition $\{S, N\backslash S\}$, we have $\sum_{i\in S} x_i \geq v(S;\{S, N\backslash S\}) \geq v(S;\{S, [N\backslash S]\})$, as externalities are positive. In particular, for $S = \{i\}$, we have $x_i \geq v(i;[N]), i = 1, \dots, n$. Furthermore, because $(x_1, \dots, x_n)$, by supposition, does not belong to the strong-core, there must exist a partition $P = \{S_1, \dots, S_m\} \neq [N]$ such that $v(S_i;P) > \sum_{j\in S_i} x_j$ for all $S_i \in P$ with $|S_i| > 1$. Then, $v(S_i;P') > \sum_{j\in S_i} x_j$, where $P' = \{S_i, N\backslash S_i\}$, because externalities are positive. But this contradicts that $(x_1, \dots, x_n)$ belongs to the δ-core. Hence our supposition is wrong and, therefore, every δ-core payoff vector $(x_1, \dots, x_n)$ belongs to the strong-core. This proves that δ-core $\subset$ strong-core.

Second, if $(x_1, \dots, x_n)$ belongs to the strong-core, then, by definition, $\sum_{i\in S} x_i \geq v(S;[N\backslash S])$ for all non-singleton coalitions$S$, and for the finest partition $P = [N]$, we have $x_i \geq v(i;[N]), i = 1, \dots, n$. Thus, every strong-core payoff vector $(x_1, \dots, x_n)$ belongs to the γ-core. This proves strong-core $\subset$ γ-core and, therefore, part (a) of the proposition.

(b) First, let $(x_1, \dots, x_n)$ be a γ-core payoff vector of a partition function game (N, v) with negative externalities. We claim that $(x_1, \dots, x_n)$ also belongs to the strong-core. Suppose not. Since $(x_1, \dots, x_n)$ belongs to the γ-core, for every partition $\{S, [N\backslash S]\}, S \subset N$, we have $\sum_{i\in S} x_i \geq v(S;\{S, [N\backslash S]\})$ and $x_i \geq v(i;[N]), i = 1, \dots, n$.

[7] See Yi (1997), Maskin (2003), and Hafalir (2007) among others for the definition.

Then, because $(x_1, \dots, x_n)$, by supposition, does not belong to the strong-core, there must be a partition $P = \{S_1, \dots, S_m\} \neq [N]$ such that $v(S_i;P) > \sum_{j \in S_i} x_j$ for all $S_i \in P$ with $|S_i| > 1$. Let $P' = \{S_i, [N \backslash S_i]\}$ denote the partition in which all but coalition S_i is a singleton. Then, since the game (N, v) has negative externalities, $v(S_i;\{S_i, [N \backslash S_i]\}) \geq v(S_i;P) > \sum_{j \in S_i} x_j$. But this contradicts that $(x_1, \dots, x_n)$ is a γ-core payoff vector. Hence, our supposition is wrong and each γ-core payoff vector $(x_1, \dots, x_n)$ also belongs to the strong-core. This proves that γ-core $\subset$ strong-core. Since, as Proposition 1 shows, strong-core $\subset$ γ-core in general. Therefore, for games with negative externalities, strong-core $=$ γ-core.

Second, suppose contrary to the assertion that for a partition function game (N, v) with negative externalities, a strong payoff vector $(x_1, \dots, x_n)$ does not belong to the δ-core. Then, we must have $\sum_{i \in S} x_i < v(S;\{S, N \backslash S\})$ for some $S \subset N$. As externalities are negative, the inequality implies that $\sum_{i \in S} x_i < v(S;\{S, [N \backslash S]\})$ for some $S \subset N$. But this contradicts that $(x_1, \dots, x_n)$ belongs to the strong-core. Thus our supposition is wrong and every strong-core payoff vector $(x_1, \dots, x_n)$ also belongs to the δ-core. This proves strong-core $\subset$ δ-core and, therefore, part (b) of the proposition. ■

Examples can be constructed to show that both inclusions in part (a) of Proposition 2 are strict in the *same* game, i.e., there are games with positive externalities in which the strong-core is not equal to either the γ- or the δ-core, but "sits" nicely between them.

Proposition 2 has important implications regarding the existence of a nonempty strong-core because it implies that for the general class of games with positive externalities, the strong-core could be nonempty even if the δ-core is empty and the strong-core is nonempty if the δ-core is nonempty. And for the general class of games with negative externalities, the strong-core is nonempty if and only if the γ-core is nonempty. This means that for at least the general class of partition function games with positive or negative games we can use the same techniques to prove the existence of a nonempty strong core as are used for proving the existence of a nonempty δ- or γ-core.

2.4 An additional relationship

The proof of Proposition 1 uses an example of a partition function game with five players. But why five, and not three or four? This is because, as will be shown, five is the minimum number of players for which the strong-core is not equal to the γ -core. We need the following definition, first introduced in Chander (2018a).

Definition 4 A partition function game (N, v) is partially super additive if for every partition $P = \{S_1, \dots, S_m\}$ with $|S_i| \geq 2, i = 1, \dots, k$, and $|S_j| = 1, j = k+1, \dots, m, k \leq m$, $\sum_{i=1}^{k} v(S_i;P) \leq v(S;P')$ where $P' = P \backslash \{S_1, \dots, S_k\} \cup \{\cup_{i=1}^{k} S_i\}$.

As Chander (2018a) notes, partial super-additivity is weaker than the familiar notion of super-additivity, which requires that combining any number of not necessarily non-singleton coalitions in a partition increases their worth. Whereas partial super-additivity requires that combining only *all* non-singleton coalitions in a partition increases their worth.

Proposition 3 *If* (N, v) *is a partially super-additive partition function game, the strong-core is equal to the* γ*-core* .

Proof By Proposition 1, the strong-core is generally a subset of the γ-core. Thus, we only need to prove that if (N, v) is partially super-additive, then every γ-core payoff vector is also a strong-core payoff vector,.

Let $(x_1, \dots, x_n)$ be a γ-core payoff vector and $P = \{S_1, \dots, S_m\}$ a partition of N. If $P = \{S_1, \dots, S_m\} \neq [N]$, then let $|S_i| > 1$, for $i = 1, \dots, k$ and $|S_j| = 1$ for $j = k + 1, \dots, m, k \leq m$, and $S = \cup_{i=1}^{k} S_i$. Since v is partially super-additive, $\sum_{i=1}^{k} v(S_i;P) \leq v(S;P')$ where $P' = P \backslash \{S_1, \dots, S_k\} \cup \{S\}$. Clearly, $P' = \{S, [N \backslash S]\}$. Since $(x_1, \dots, x_n)$ is a γ-core payoff vector, $\sum_{i \in S} x_i \geq v(S;\{S, [N \backslash S]\}) = v(S;P') \geq \sum_{i=1}^{k} v(S_i;P)$, because v is partially super additive. This inequality can be rewritten as $\sum_{i=1}^{k} \sum_{j \in S_i} x_j \geq \sum_{i=1}^{k} v(S_i;P)$ and, therefore, $\sum_{j \in S_i} x_j \geq v(S_i;P)$ for at least one $S_i \in \{S_1, \dots, S_k\} \subset P$, i.e., for at least one coalition $S_i \in P$ with $|S_i| > 1$, we have $\sum_{j \in S_i} x_j \geq v(S_i;P)$. Thus, $(x_1, \dots, x_n)$ is a strong-core payoff vector.

If $(x_1, \dots, x_n)$ is a γ-core payoff vector and $P = [N]$, then $x_i \geq v(i;\{i, [N \backslash i]\}) = v(i;[N])$. Thus, in this case too $(x_1, \dots, x_n)$ is a strong-core payoff vector. ■

Partial super-additivity is trivially satisfied in all 3-player partition function games. It is also satisfied in those 4-player partition function games (N, v), including the Cournot oligopoly modelled as a partition function game, in which the grand coalition is an efficient coalition structure, i.e. $v(N;N) \geq \sum_{S \in P} v(S;P)$ for any partition P.Thus, if the grand coalition is an efficient coalition structure, the strong-core is equal to the γ-core in all 3 and 4 player games. In other words, if the grand coalition is an efficient coalition structure, then the strong-core is not necessarily equal to the γ-core *only* in games with five or more players. For this reason, the proof of Proposition 1 uses an example with five players.

2.5 A class of games with nonempty strong-cores

As we study below the Cournot oligopoly with symmetric firms, it is useful to first prove existence of a nonempty strong-core for a general class of symmetric partition function games in which larger coalitions in each partition have lower per-member payoffs (see e.g. Ray and Vohra 1997; Yi 1997; Funaki and Yamato 1999; Chander 2007).[8] The proof is constructive, as it is shown that a specific feasible payoff vector belongs to the strong-core.

[8] Although the γ-cores of these games are known to be non-empty (Chander 2018a), it does not follow that their strong-cores are also non-empty, because these games may neither be partially super-additive nor exhibit negative externalities.

Proposition 4 *Let* (N, v) *be a symmetric partition function game such that for every partition* $P = \{S_1, \ldots, S_m\}$, $v(S_i;P)/|S_i| < (=) v(S_j;P)/|S_j|$ *if* $|S_i| > (=) |S_j|$, $i,j \in \{1, \ldots, m\}$ *and* $v(N;N) \geq \sum_{S_i \in P} v(S_i;P)$. *Then, the strong-core of* (N, v) *is non-empty.*

Proof Let $(x_1, \ldots, x_n)$ be the feasible payoff vector with equal shares, i.e., $\sum_{i \in N} x_i = v(N;N)$ and $x_i = x_j, i,j \in N$. We claim that $(x_1, \ldots, x_n)$ is a strong-core payoff vector.

Let $P = \{S_1, \ldots, S_m\} \neq N$ be some partition of N. If $P = [N]$, then $x_i \geq v(i;[N])$ for all $\{i\} \in [N]$, because $v(N;N) \geq \sum_{i \in N} v(i;[N])$, $\sum_{i \in N} x_i = v(N;N)$, $v(i;[N]) = v(j;[N])$, and $x_i = x_j$ for all $i,j \in N$. If $P \neq [N]$, then the number of coalitions in the partition is $m \geq 2, m < n$. Without loss of generality assume $|S_1| \geq |S_2| \geq \cdots \geq |S_m|$. Thus, $n > m \geq 2$ and $\sum_{i=1}^{m} v(S_i;P) \leq v(N;N) = \sum_{i \in N} x_i$, as hypothesized. This inequality implies $v(S_1;P) \leq \sum_{i \in S_1} x_i$, since $v(S_1;P)/|S_1| \leq v(S_j;P)/|S_j|$ for all $S_j \in P$ and $x_i = x_j, i,j \in N$. Since $n \geq 3$ and $P \neq [N], N$, we must have $|S_1| \geq 2$. This proves that each partition $P \neq [N]$, includes at least one non-singleton coalition which is worse-off relative to the feasible payoff vector with equal shares $(x_1, \ldots, x_n)$ and $x_i \geq v(i;[N]), i = 1, \ldots, n$. Thus, by definition of the strong-core, the payoff vector with equal shares $(x_1, \ldots, x_n)$ belongs to the strong-core. ■

3 The Cournot oligopoly and the strong-core

I formulate Cournot oligopoly first as a strategic game and then, as in the approach pioneered by Ichiishi (1981) and Zhao (1996), I convert the strategic game into a partition function game by proving existence of a unique Nash equilibrium for each induced strategic game in which each coalition in a partition acts as one single player and the worth/payoff of each coalition in the partition is equal to the total profit of the coalition in the Nash equilibrium of the induced strategic game. In this context, a singleton coalition in a partition represents a standalone/independent firm and a non-singleton coalition a cartel, a partition an industry structure, and a partition function a compilation of information about the profits earned by the firms in every possible industry structure. The strong-core of a Cournot oligopoly modelled as a partition function game enables us to test stability of the merger-to-monopoly against *every* possible industry structure, whereas the familiar γ- and δ-cores enable us to test stability of the merger-to-monopoly only against certain subsets of industry structures.

The set of oligopolistic firms is $N = \{1, \ldots, n\}$. Let $p(q)$ denote the inverse demand function faced by these firms, where q is the total demand. I assume that the inverse demand function is differentiable and strictly decreasing and concave, i.e., $p'(q) < 0$ and $p''(q) \leq 0$. These assumptions are satisfied by the linear demand function $p(q) = a - bq, a, b > 0$, and imply that the revenue function $p(q)q_i$ of each firm i is concave in outputs $q_1, \ldots, q_n$ of the n firms, i.e. the marginal revenue $p(q) + p'(q)q_i$

of each firm i is non-increasing in $q_1, \ldots, q_n$, because $p'(q) + p''(q)q_i \leq 0$ for each fixed $q_i \geq 0$ and $2p'(q) + p''(q)q_i \leq 0$ for each fixed $q_j \geq 0, j \neq i$.

The cost function of each firm i is $c_i(q_i)$ with $c_i(0) = 0$. I assume that the cost function of each firm is differentiable, strictly increasing and strictly convex, i.e., $c_i'(0) = 0$, $c_i'\left(q_i\right) > 0, q_i > 0$, and $c_i''\left(q_i\right) > 0, q_i \geq 0$. Strictly convex cost functions imply that the same output can be produced by a coalition of two or more firms at a lower cost than by a single, standalone firm. Thus, strictly convex cost functions imply an additional incentive for the oligopolistic firms to merge. Although, the results below also hold for constant marginal cost functions, i.e., for the cost functions $c_i q_i, c_i > 0$, but strictly convex functions lead to a more concrete analysis by avoiding multiple equilibria.[9] For the time being, I do not assume capacity constraints on outputs of the firms. Instead, I assume that the firms can expand their production capacities as and when necessary. This leads to a long-run equilibrium analysis of Cournot oligopoly.

The profit function of each firm i is $\pi_i\left(q_1, \ldots, q_n\right) = p(q)q_i - c_i\left(q_i\right)$, where $q = \sum_{j \in N} q_j$. It may be noted that, by definition of the profit function, the profit of a firm is maximized only if its cost of producing the profit maximizing output is minimized. In order to avoid occurrence of a corner solution (i.e. zero output), I assume that there exists an upper bound q^0 such that $p\left(q^0\right) + p'\left(q^0\right)q^0 - c_i'\left(q^0\right) < 0$ for each firm i. As both $p(q)$ and $p'(q)$ are non-increasing functions of q, this assumption implies that a standalone profit maximizing firm or a firm as a member of a profit maximizing coalition will never produce an output that is larger than q^0 even if it has the capacity to do so. I assume further that $p\left((n-1)q^0\right) > 0$. As $c_i'(0) = 0$, this additional assumption implies that a standalone profit maximizing firm or a firm as a member of a profit maximizing coalition will always produce a positive amount, irrespective of the output of other profit maximizing firms or coalitions.

3.1 The oligopoly game

Let $A_i = \left[0, q^0\right], A = A_1 \times \cdots \times A_n$, and $\pi = \left(\pi_1, \ldots, \pi_n\right)$. I shall refer to the strategic game (N, A, π) as the *oligopoly game*. Clearly, each strategy set A_i is compact and convex and each π_i is concave and continuous in $q_1, \ldots, q_n$.

Lemma *The oligopoly game* (N, A, π) *admits a unique Nash equilibrium* $(\bar{q}_1, \ldots, \bar{q}_n)$ ·

The proof of the Lemma is in the "Appendix". I interpret the Nash equilibrium of the oligopoly game as the *status quo outcome* and partition $[N]$ as the *status quo industry structure* as they describe the situation before any mergers between the oligopolistic firms. I next define the payoffs of coalitions in every possible industry structure if the firms are free to merge.

[9] Some authors consider constant marginal costs a relatively less interesting case (see e.g. Perry and Porter 1985).

Let $\left(N^P, A^P, \pi^P\right)$ denote the induced game when the firms form a partition $P = \{S_1, \dots, S_m\}$ and each coalition $S_i, i = 1, \dots, m$, in the partition acts as one single player. Since each $\pi_i\left(q_1, \dots, q_n\right)$ is concave and continuous in $q_1, \dots, q_n$, the payoff function $\pi^P_{S_i}\left(q_1, \dots, q_n\right) \equiv \sum_{j \in S_i} \pi_j(q_1, \dots, q_n))$ of coalition S_i in partition P is also concave and continuous in $q_1, \dots, q_n$. Moreover, the strategy set $\times_{j \in S_i} A_j$ of coalition S_i is compact and convex. Therefore, as in the Lemma above, the induced game $\left(N^P, A^P, \pi^P\right)$ also admits a unique Nash equilibrium $\left(\bar{q}^P_1, \dots, \bar{q}^P_n\right)$. Let $v(S_i;P)$ be equal to the Nash equilibrium payoff of coalition S_i in the induced game $\left(N^P, A^P, \pi^P\right)$, i.e. $v\left(S_i;P\right) = \sum_{j \in S_i} \pi_j(\bar{q}^P_1, \dots, \bar{q}^P_n)$. Then, (N, v) is the partition function game form/representation of the oligopoly game (N, A, π). In this partition function game, the grand coalition is efficient, because the grand coalition can choose at least the same outputs as the coalitions in any partition and, thus, the grand coalition can obtain at least as much profit as the total profit of all firms in any partition, i.e. $v(N;N) \geq \sum_{S_i \in P} v(S_i;P)$. As all firms in an oligopoly face the same demand function $p(.)$, the firms are identical if their cost functions are identical, i.e. $c_i(.) = c_j(.), i, j \in N$.

Proposition 5 *If* (N, v) *is the partition function game form of the oligopoly game* (N, A, π) *and* x *is a strong-core payoff vector, then* $x_i \geq v(i;[N]) = \pi_i(\bar{q}_1, \dots, \bar{q}_n)$ *for each* $i \in N$, *where* $(\bar{q}_1, \dots, \bar{q}_n)$ *is the Nash equilibrium of the oligopoly game* (N, A, π).

Proof Since, as hypothesized, x is a strong-core payoff vector in the partition function game (N, v), we have $x_i \geq v(i;\{i, [N \backslash i]\})$ for each i and partition $\{i, [N \backslash i]\} \ni i$. Therefore, $x_i \geq v(i;[N])$ for each i, because $\left\{i; \left[N \backslash i\right]\right\} = [N]$ for each $i \in N$. By definition of v, we have $v(i;\{i, [N \backslash i]\}) = \pi_i\left(\bar{q}_1, \dots, \bar{q}_n\right)$. Thus, $x_i \geq v(i;[N]) = \pi_i\left(\bar{q}_1, \dots, \bar{q}_n\right)$. □

Proposition 5 implies that oligopolistic firms will have incentives to merge from the status quo industry structure to monopoly if the distribution of monopoly profits is a strong-core payoff vector.[10] In other words, if the strong-core of the oligopoly game is nonempty and distribution of monopoly profits is a strong-core payoff vector, then not only every oligopolistic firm has incentives to merge from the status quo industry structure to monopoly, but also if the firms indeed merge to monopoly, the merger-to-monopoly is stable against every possible industry structure. We confirm below that the strong-core is indeed nonempty both in the long- and short-run models of Cournot oligopoly.

It may be noted that Proposition 5 does not show that oligopolistic firms will have incentives to merge to monopoly from an industry structure that is different from the status quo industry structure. The firms, in fact, may have no such incentives even if the distribution of monopoly profits is a strong-core payoff, because for any strong-core payoff vector x and partition $P \neq [N], N$, we may have $\sum_{i \in S_k} x_i \leq v(S_k, P)$ for at

[10] The oligopolistic firms may not have the same incentives to merge to monopoly if the distribution of monopoly profits is not a strong-core payoff vector, but instead is, say, a δ-core payoff vector.

least one non-singleton coalition $S_k \in P$, i.e., at least one non-singleton coalition in any industry structure that is different from the status quo industry structure may be worse-off if the firms merge to monopoly. This means mergers among few, but not all, oligopolistic firms can adversely affect their incentives to merge to monopoly.

Proposition 6 *If all oligopolistic firms are identical, i.e.*$c_i(.) = c_j(.), i, j \in N$, *then the partition function game form* (N, v) *of the oligopoly game* (N, A, π) *is symmetric and the strong-core is nonempty.*

Proof For each partition $P = \{S_1, \dots, S_m\}$, let $(\bar{q}_1^P, \dots, \bar{q}_n^P)$ denote the unique Nash equilibrium of the induced game $\left(N^P, A^P, \pi^P\right)$. The first order conditions for profit maximization by coalition $S_k \in P$ are $c_i'\left(\bar{q}_i^P\right) = p\left(\sum_{i\in N}\bar{q}_i^P\right) + \left(\sum_{j\in S_k}\bar{q}_j^P\right) p'\left(\sum_{i\in N}\bar{q}_i^P\right)$, $i \in S_k$. As each c_i is strictly convex and $c_i(.) = c_j(.), i, j \in S_k$, the equalities imply $\bar{q}_i^P = \bar{q}_j^P, i, j \in S_k$. Because $p'(q) < 0$, $p''(q) \le 0$, each c_i is strictly convex, and $c_i(.) = c_j(.), i, j \in N$, the equalities imply $\bar{q}_j^P > (=)\bar{q}_i^P$ if $j \in S_k \in P, i \in S_r \in P$ and $\left|S_k\right| < (=)\left|S_r\right|$. That is, the output of each firm in a larger coalition is lower. Because all firms are identical and face the same prices,$v(S_k;P)/\left|S_k\right| > (=)v(S_r;P)/\left|S_r\right|$ if $\left|S_k\right| < (=)\left|S_r\right|, k, r \in \{1, \dots, m\}$. Furthermore, for each partition $P = \left\{S_1, \dots, S_m\right\}$, we have $v(N;N) \ge \sum_{k=1}^{m} v\left(S_k;P\right)$, because the firms as members of the grand coalition can at least choose the same outputs as members of the various coalitions in partition P. The proof now follows from Proposition 4. □

Rajan (1989) proves existence of a nonempty γ-core for the Cournot oligopoly with four identical firms. Proposition 6 is more general than Rajan's result because the number of firms is not restricted to four. It can be equal to any finite number and such that the strong-core is a subset of the γ-core. We show that for the oligopoly game it may actually be a strict subset.

Example 1 Let $N = \{1,2,3,4,5\}$, $p(q) = 1 - q$ and $c_i\left(q_i\right) = \frac{1}{2}q_i^2, i \in N$.

The example is a specific case of the oligopoly model above. It has no less than five firms/players because for games in which the grand coalition is an efficient partition, the strong-core, as pointed out above, cannnot be a strict subset of the γ-core if the number of players is less than five. The oligopoly game is known to be a game with positive, not negative, externalities (see e.g. Yi 1997; Kóczy 2018). We show that the game is also not partially super-additive. Thus, none of the conditions for equality of the strong-core and the γ-core are satisfied.

Let $P = \{\{1\}, \{2,3\}, \{4,5\}\}$ and $H = \{\{1\}, \{2,3,4,5\}\}$. We claim that $v(\{2,3,4,5\};H) < v(\{2,3\};P) + v(\{4,5\};P)$. It is easily seen that $\bar{q}_1^P = \frac{3}{17}, \bar{q}_2^P = \bar{q}_3^P = \bar{q}_4^P = \bar{q}_5^P = \frac{2}{17}$ and, thus, $v(\{2,3\};P) = v(\{4,5\};P) = 5\ \left(\frac{2}{17}\right)^2$. Similarly, $\bar{q}_1^H = \frac{5}{23}, \bar{q}_2^H = \bar{q}_3^H = \bar{q}_4^H = \bar{q}_5^H = \frac{2}{23}$ and $v(\{2,3,4,5\};H) = 18\left(\frac{2}{23}\right)^2 < v(\{2,3\};P) + v(\{3,4\};P) = 5\ \left(\frac{2}{17}\right)^2 + 5\left(\frac{2}{17}\right)^2$.

It is noteworthy that $\bar{q}_1^H = \frac{5}{23} > \bar{q}_1^P = \frac{3}{17}$, but $\bar{q}_2^H = \bar{q}_3^H = \bar{q}_4^H = \bar{q}_5^H = \frac{2}{23} < \bar{q}_1^P = \frac{3}{17}$, $\bar{q}_2^P = \bar{q}_3^P = \bar{q}_4^P = \bar{q}_5^P = \frac{2}{17}$. That is, if the two cartels $\{2,3\}$ and $\{4,5\}$ merge, then each firm in these cartels produces a lower output, but the outside firm $\{1\}$ produces a higher output. That is because each firm in the merged cartels must take into account the effect of its production not only on its own profit, but also on the profits of all other firms in the merged cartels. But a decrease in the outputs of the firms in the merged cartels makes it profitable for the independent firm outside the cartels to raise its output.[11]

I now show that the strong-core of the oligopoly game in Example 1 is a strict subset of the γ-core. Because the strong-core is generally a subset of the γ-core, as Proposition 1 shows, it is sufficient to identify a γ-core payoff vector that does not belong to the strong-core. To that end, note that $v(N;N) = \frac{5}{22}$. Let $x_1 = v(N;\{N\}) - v(\{2,3,4,5\};H) = \frac{5}{22} - 18(\frac{2}{23})^2$ and $x_2 = x_3 = x_4 = x_5 = \frac{1}{4}v(\{2,3,4,5\};H) = \frac{18}{4}(\frac{2}{23})^2$. Then, the feasible payoff vector $(x_1, \dots, x_5)$ belongs to the γ-core, because the game is symmetric and $\frac{1}{4}v(\{2,3,4,5\};H) > \frac{1}{3}v(\{3,4,5\};\{\{1\},\{2\},\{3,4,5\}\}) > \frac{1}{2}v(\{4,5\};\{\{1\},\{2\},\{3\},\{4,5\}) > v(\{i\};[N])$ and $x_1 > \frac{1}{4}v(\{2,3,4,5\};H)$. But the feasible payoff vector $(x_1, \dots, x_5)$ does not belong to the strong-core, because $x_2 + x_3 = x_4 + x_5 = \frac{18}{2}(\frac{2}{23})^2 < v(\{2,3\};P) = v(\{4,5\};P) = 5(\frac{2}{17})^2$.

3.2 A short-run model of *Cournot oligopoly*

A number of studies (e.g. Radner et al. 2001; Yong 2004) have focused on Cournot oligopoly of firms with fixed production capacities. Radner et al. (2001) proves existence of a non-empty α-core, Zhao (1999) of a non-empty β-core, and Lardon (2012) of a non-empty γ-core for a Cournot oligopoly of firms with fixed production capacities. These studies assume *arbitrary* production capacities and do not take into account the effect that the existing industry structure and production capacities may have on the incentives of oligopolistic firms to merge. It makes little sense to study stability of the merger-to-monopoly if the existing industry structure and production capacities are such that the oligopolistic firms do not have incentives to merge to monopoly. As made clear above, oligopolistic firms do have incentives to merge to monopoly from the status quo industry structure when their outputs are equal to their Nash equilibrium outputs, but not necessarily from other industry structures and outputs that are not equal to their Nash equilibrium outputs. Thus, for a short-run analysis of Cournot oligopoly, I study stability of the merger-to-monopoly only in the case of the status quo industry structure and production capacities that are equal to the Nash equilibrium outputs of the firms in the status quo industry structure. I shall refer to these production capacities as the "historical" production capacities, as they are equal to the equilibrium outputs of the firms in the status quo industry structure.

I assume that in the short-run no firm can expand its current production capacity and the production capacities of the firms cannot be transferred or pooled because of either plant location or incompatibility of technologies. This means no firm in

[11] In public good models this phenomenon is known as "free riding" and in climate change models as "leakages".

a cartel can use the production capacity of another firm in the cartel; the firms in a cartel can only coordinate their production decisions to maximize their joint profits. In sum, the output of no firm in the short-run can be larger than its historical production capacity and the firms may form a cartel/coalition starting from the status quo industry structure purely for strategic reasons, and not for obtaining some technological advantage.

Formally, the strategy set of firm i is now $\bar{A}_i = \left[0, \bar{q}_i\right], i \in N$, and $\bar{A} = \bar{A}_1 \times \cdots \times \bar{A}_n$. I shall refer to the strategic game $(N, \bar{A}, \pi)$ as the *constrained* oligopoly game. Let $\left(N^P, \bar{A}^P, \pi^P\right)$ denote the induced game when the firms form a partition $P = \{S_1, \ldots, S_m\}$ and each coalition $S_i, i = 1, \ldots, m$, in the partition acts as one single player. Clearly, the strategy set $\times_{j \in S_i} \bar{A}_j$ of coalition S_i is compact and convex and, by the same argument as in the Lemma, each induced game $\left(N^P, \bar{A}^P, \pi^P\right)$ admits a unique Nash equilibrium $(\bar{\bar{q}}_1^P, \ldots, \bar{\bar{q}}_n^P)$. In particular, the Nash equilibrium $(\bar{q}_1, \ldots, \bar{q}_n)$ of the original oligopoly game (N, A, π) is also the unique Nash equilibrium of the constrained oligopoly game$(N, \bar{A}, \pi)$. For this reason, I will continue to refer to the Nash equilibrium outcome of the constrained oligopoly as the *status quo* outcome.

From our discussion of the proof of Proposition 5 we know that oligopolistic firms will have incentives to merge to monopoly starting from the status quo industry structure if distribution of the monopoly profits is a strong-core payoff vector. But will the merger-to-monopoly, if the firms indeed merge to monopoly, be stable or equivalently, is the strong-core of the oligopoly game $\left(N^P, \bar{A}^P, \pi^P\right)$ nonempty? To answer this question, we first prove the following result.

Proposition 7 *For each partition* $P = \left\{S, \left[N \backslash S\right]\right\}, |S| \neq 1$, *let* $(\bar{\bar{q}}_1^P, \ldots, \bar{\bar{q}}_n^P)$ *denote the unique Nash equilibrium of the induced constrained oligopoly game* $\left(N^P, \bar{A}^P, \pi^P\right)$. *Then*, (*i*) $\bar{\bar{q}}_j^P = \bar{q}_j$ *for each* $j \in N \backslash S$ *and* (*ii*) $\bar{\bar{q}}^P \equiv \sum_{i \in N} \bar{\bar{q}}_i^P < \sum_{i \in N} \bar{q}_i \equiv \bar{q}$.

Proof (i) By definition of the induced constrained game, $\bar{\bar{q}}_i^P \leq \bar{q}_i, i \in N$, and, thus, $\bar{\bar{q}}^P \leq \bar{q}$. Suppose contrary to the assertion that $\bar{\bar{q}}_j^P < \bar{q}_j$ for some $j \in N \backslash S$. Then, by the first order conditions for a Nash equilibrium, $c_j'(\bar{q}_j) = p(\bar{q}) + \bar{q}_j p'(\bar{q}) < p(\bar{q}) + \bar{\bar{q}}_j^P p'(\bar{q}) \leq p\left(\bar{\bar{q}}^P\right)$ $+\bar{\bar{q}}_j^P p'\left(\bar{\bar{q}}^P\right) = c_j'\left(\bar{\bar{q}}_j^P\right) \Rightarrow \bar{q}_j < \bar{\bar{q}}_j^P$, since c_j is strictly convex. But this contradicts our supposition that $\bar{\bar{q}}_j^P < \bar{q}_j$. Therefore, $\bar{\bar{q}}_j^P = \bar{q}_j$ for all $j \in N \backslash S$.

(ii) Suppose contrary to the assertion that $\bar{\bar{q}}^P = \bar{q}$. This is possible only if $\bar{\bar{q}}_j^P = \bar{q}_j$ for all $j \in N$, because $\bar{\bar{q}}_j^P \leq \bar{q}_j$ for all $j \in N$. Thus, for each $i \in S$, $c_i'\left(\bar{\bar{q}}_i^P\right) = (\sum_{j \in S} \bar{\bar{q}}_j^P)$ $p'\left(\bar{\bar{q}}^P\right) + p\left(\bar{\bar{q}}^P\right) = (\sum_{i \in S} \bar{q}_i) p'(\bar{q}) + p(\bar{q}) < \bar{q}_i p'(\bar{q}) + p(\bar{q}) = c_i'\left(\bar{q}_i\right) \Rightarrow \bar{q}_i > \bar{\bar{q}}_i^P$, since c_i is strictly convex. But this contradicts our supposition that $\bar{\bar{q}}^P = \bar{q}$ and $\bar{\bar{q}}_j^P = \bar{q}_j, j \in N$. Therefore, $\bar{\bar{q}}^P < \bar{q}$. □

Proposition 7 implies that if the oligopolistic firms form a cartel starting from the status quo industry structure, then the total industry output is lower compared to the total status quo output and, therefore, the market price is higher. The total profit of the firms in the cartel is not necessarily higher compared to their total *status quo* profit despite a higher market price because the total output of the cartel is lower. The profit of each standalone firm is higher even though the output of each standalone firm is the same as before, because the capacity constraint of each standalone firm is binding and the production capacities, by assumption, cannot be expanded in the short-run.

It was shown for Example 1 that the partition function game form of the Cournot oligopoly (N, A, π) is *not* partially super-additive. However, it turns out that the partition function game form of the constrained oligopoly game $\left(N^P, \bar{A}^P, \pi^P\right)$ is partially super-additive.

Proposition 8 *The partition function game form of the constrained oligopoly game* $(N, \bar{A}, \pi)$ *is partially super-additive.*

Proof As in the Lemma, for each partition $P = \{S_1, \ldots, S_m\}$, the induced game $\left(N^P, \bar{A}^P, \pi^P\right)$ admits a unique Nash equilibrium. Let $(\overline{q}_1^P, \ldots, \overline{q}_n^P)$ denote the unique Nash equilibrium. Then, by definition of the constrained oligopoly game, $\overline{q}^P \equiv \sum_{i \in N} \overline{q}_i^P \leq \bar{q} = \sum_{i \in N} \bar{q}_i = \sum_{i \in N} k_i$. I claim that, as in Proposition 7, $\overline{q}_j^P = \bar{q}_j = k_i$ for each standalone firm j in partition P. Suppose contrary to the claim that $\overline{q}_j^P < \bar{q}_j$ for some singleton coalition $\{j\} \in P$. Then, $c_j'\left(\overline{q}_j^P\right) = p\left(\overline{q}^P\right) + \overline{q}_j^P p'\left(\overline{q}^P\right) \geq p(\bar{q}) + \overline{q}_j^P p'(\bar{q}) > p(\bar{q}) + \bar{q}_j p'(\bar{q}) = c_j'\left(\bar{q}_j\right) \Rightarrow \bar{q}_j < \overline{q}_j^P$, because c_j is strictly convex. But this contradicts our supposition that $\overline{q}_j^P < \bar{q}_j$. Thus, $\overline{q}_j^P = \bar{q}_j = k_j$ for each standalone firm j in P.

Let $\bar{v}\left(S_i, P\right) = \sum_{j \in S_i} \pi_j \left(\overline{q}_1^P, \ldots, \overline{q}_n^P\right)$. Without loss of generality assume that in partition$P = \{S_1, \ldots, S_m\}$, $\left|S_i\right| > 1, i = 1, \ldots, r,$ and $\left|S_j\right| = 1, j = r+1, \ldots, m.$ Let $S = \cup_{i=1}^r S_i$, $H = P \backslash \left\{S_1, \ldots, S_r\right\} \cup \{S\} = \left\{S, \left[N \backslash S\right]\right\}$ and let $(\overline{q}_1^H, \ldots, \overline{q}_n^H)$ be the unique Nash equilibrium of the induced game $\left(N^H, \bar{A}^H, \pi^H\right)$ and $\bar{v}(S;H) = \sum_{i \in S} \pi_i (\overline{q}_1^H, \ldots, \overline{q}_n^H)$. We claim that$\sum_{i=1}^r \bar{v}\left(S_i;P\right) \leq \bar{v}(S;H)$, that is, $\bar{v}$ is partially superadditive. We first prove that $\overline{q}^H \equiv \sum_{i \in N} \overline{q}_i^H \leq \sum_{i \in N} \overline{q}_i^P = \overline{q}^P$. Suppose contrary to the assertion that $\overline{q}^H > \overline{q}^P$. Then, since $\overline{q}_j^H \leq \overline{q}_j^P = k_i$ for each $j \in N \backslash S$, $\sum_{i \in S} \overline{q}_i^H > \sum_{i \in S} \overline{q}_i^P$ and, therefore, $\sum_{j \in S_i} \overline{q}_j^H > \sum_{j \in S_i} \overline{q}_j^P$ for at least one coalition S_i, and for each $j \in S_i, c_j'\left(\overline{q}_j^P\right) = (\sum_{k \in S_i} \overline{q}_k^P) p'\left(\overline{q}^P\right) + p(\overline{q}^P) > \left(\sum_{k \in S_i} \overline{q}_k^H\right) p'\left(\overline{q}^P\right)$ $+ p\left(\overline{q}^P\right) \geq (\sum_{k \in S_i} \overline{q}_k^H) p'\left(\overline{q}^H\right) + p\left(\overline{q}^H\right) = c_j'\left(\overline{q}_j^H\right)$. Thus,$\overline{q}_j^P > \overline{q}_j^H$, since c_i is strictly convex. But this contradicts that $\sum_{j \in S_i} \overline{q}_j^H > \sum_{j \in S_i} \overline{q}_j^P$. Hence,$\sum_{i \in S} \overline{q}_i^H \leq \sum_{i \in S} \overline{q}_i^P$

and $\overline{q}^H \leq \overline{q}^P$. This also implies that $\overline{q}_j^H = \bar{q}_j = k_j$ for all $j \in N \backslash S$. To see this, suppose on the contrary that $\overline{q}_j^H < \bar{q}_j$ for some $j \in N \backslash S$. Then, since $\overline{q}^H \leq \overline{q}^P \leq \bar{q}$ (as shown), $c_j'\left(\overline{q}_j^H\right) = p\left(\overline{q}^H\right) + \overline{q}_j^H p'\left(\overline{q}^H\right) \geq p(\bar{q}) + \overline{q}_j^H p'(\bar{q}) > p(\bar{q}) + \bar{q}_j p'(\bar{q}) = c_j'\left(\bar{q}_j\right) \Rightarrow \bar{q}_j < \overline{q}_j^H$, which contradicts our supposition that $\overline{q}_j^H < \bar{q}_j$. Therefore, $\overline{q}_j^H = \bar{q}_j = k_i$ for all $j \in N \backslash S$. Since coalition S could have chosen for each $i \in S$ an output level $\overline{q}_i^P$ but chose instead $\overline{q}_i^H$ and $\overline{q}_j^H = \overline{q}_j^P = \bar{q}_j = k_j$ for each $j \in N \backslash S$, it follows that $\bar{v}(S;H) = \sum_{i \in S} \pi_i(\overline{q}_1^H, \ldots, \overline{q}_n^H) \geq \sum_{i=1}^r \sum_{j \in S_i} \pi_j\left(\overline{q}_1^P, \ldots, \overline{q}_n^P\right) = \sum_{i=1}^r \bar{v}\left(S_i, P\right)$, i.e. the partition function game $(N, \bar{v})$ is partially super-additive. □

Proposition 8 and its proof provide useful insights into the industry structure that may emerge in the short-run if oligopolistic firms were allowed to merge starting from the status quo industry structure and if the firms were not able to expand their existing/historical production capacities. Proposition 8 implies that the firms may form at most *one* cartel followed possibly by a fringe of standalone firms, because the game is partially super-additive and, therefore, forming just one rather than two or more cartels is more profitable for the firms in the cartels. If the firms indeed form a cartel, then, as seen from the discussion of Proposition 5, the firms may not have incentives to merge-to-monopoly.

It is noteworthy that Propositions 7 and 8, unlike Proposition 6, hold even if the firms are not necessarily identical. Propositions 8 and 3 imply that the strong-core of the constrained oligopoly is equal to its γ-core. Therefore, the strong-core of the constrained oligopoly is nonempty, because, as Lardon (2012) shows, the γ-core of the constrained oligopoly is nonempty. This means the firms in the Cournot oligopoly with historical capacity constraints will have incentives to merge from the status quo industry structure to monopoly if they cannot expand their production capacities.

4 Conclusion

From a game theory perspective, the paper has motivated and introduced a new core concept for partition function games. The new core concept, unlike the previous core concepts, makes no ad hoc assumption regarding market's ethos, i.e. the partition that may be formed by the players subsequent to a deviation. This seems to settle a long standing debate on which core concept to use.

The strong-core offers a view on the stability of cartels, especially on the stability of the merger-to-monopoly, that is independent of market's ethos.[12] It was shown that firms in Cournot oligopoly not only have incentives to merge to monopoly but the merger-to-monopoly is stable, both in the long- and short-run.

[12] The strong-core also has other applications including to models of climate change (see e.g. Chander 2018a,b; Helm 2001).

Several assumptions limit our analysis. We assumed a fixed number of oligopolistic firms. Although this is a standard assumption of the Cournot model, threat of entry in the real-world can weaken the incentives of farsighted firms to merge to monopoly. Except for a brief discussion concerning the interpretation of the strong-core of the Cournot oligopoly, the paper leaves unspecified details of the negotiation or merger process among oligopolistic firms. Real-world mergers typically involve only two firms at a time, even when larger mergers are theoretically possible. This can affect the incentives of the firms to merge to monopoly, because, as shown in this paper, the firms may not have incentives to merge to monopoly starting from an industry structure that is different from the status quo industry structure. For this reason real-world merger processes may get stuck in an industry structure consisting of a cartel followed by a fringe of standalone firms (because, as shown, forming just one rather than two or more cartels is more profitable for the firms in the cartels) and never converge to monopoly, even in the absence of an antitrust policy.

Appendix

Proof of the Lemma Since each $\pi_i(.)$ is concave and continuous in $q_1, \ldots, q_n$ and each A_i is compact and convex, the game (N, A, π) admits a Nash equilibrium $(\bar{q}_1, \ldots, \bar{q}_n)$. Suppose contrary to the assertion that the game has another Nash equilibrium, say $(\bar{\bar{q}}_1, \ldots, \bar{\bar{q}}_n)$, and $(\bar{q}_1, \ldots, \bar{q}_n) \neq (\bar{\bar{q}}_1, \ldots, \bar{\bar{q}}_n)$. Without loss of generality, let $\bar{q} = \sum_{i \in N} \bar{q}_i \geq \sum_{i \in N} \bar{\bar{q}}_i = \bar{\bar{q}}$. Since $(\bar{q}_1, \ldots, \bar{q}_n) \neq (\bar{\bar{q}}_1, \ldots, \bar{\bar{q}}_n)$, $\bar{q}_i > \bar{\bar{q}}_i$ for at least one i. Furthermore, $p'\left(\bar{\bar{q}}\right)\bar{\bar{q}}_i + p\left(\bar{\bar{q}}\right) > p'\left(\bar{\bar{q}}\right)\bar{q}_i + p\left(\bar{\bar{q}}\right) \geq p'(\bar{q})\bar{q}_i + p(\bar{q})$, since $\bar{q} \geq \bar{\bar{q}}$ and by assumption the marginal revenue of each firm is non-increasing with total demand q. From the first order conditions for a Nash equilibrium $c_i'\left(\bar{\bar{q}}_i\right) = p'\left(\bar{\bar{q}}\right)\bar{\bar{q}}_i + p\left(\bar{\bar{q}}\right) > p'(\bar{q})\bar{q}_i + p(\bar{q}) = c_i'(\bar{q}_i)$ implying $\bar{q}_i < \bar{\bar{q}}_i$, which is a contradiction. □

References

Aumann RJ (1961) The core of a cooperative game without side payments. Trans Math Soc 41:539–552

Chander P (2007) The gamma-core and coalition formation. Int J Game Theory 2007:539–556

Chander P (2018a) The core of a strategic game. BE J Theor Econ 19(1):2017–2155

Chander P (2018b) Game theory and climate change. Columbia University Press, New York

Chander P, Tulkens H (1997) The core of an economy with multilateral environmental externalities. Int J Game Theory 26:379–401

Corchòn LC, Marini MA (2018) Handbook of game theory and industrial organization, vol 1. Edward Elgar Publishing, Cheltenham

Deneckere R, Davidson C (1985) Incentives to form coalitions with Bertrand competition. Rand J Econ 16:473–486

Farrell J, Shapiro C (1990) Horizontal mergers: an equilibrium analysis. Am Econ Rev 80:107–126

Funaki Y, Yamato T (1999) The core of an economy with a common pool resource: a partition function form approach. Int J Game Theory 28:157–171

Gillies DB (1953) Discriminatory and bargaining solutions to a class of symmetric n-person games. In: Kuhn HW, Tucker AW (eds) Contributions to theory of games 2. Princeton University Press, Princeton, pp 325–342
Hafalir IE (2007) Efficiency in coalitional games with externalities. Games Econ Behav 61:242–258
Hart S, Kurz M (1983) Endogenous formation of coalitions. Econometrica 51:1047–1064
Helm C (2001) On the existence of a cooperative solution for a coalitional game with externalities. Int J Game Theory 30:141–147
Huang C-Y, Sjostrom T (2003) Consistent solutions for cooperative games with externalities. Games Econ Behav 43:196–213
Ichiishi T (1981) A social equilibrium existence lemma. Econometrica 49:369–377
Jaskow PL (1975) Firm decision-making processes and oligopolistic theory. Am Econ Rev 65(2):270–279
Kóczy LÁ (2007) A recursive core for partition function form games. Theor Decis 63:41–51
Kóczy LÁ (2018) Partition function form games, theory and decision library C 48. Springer International Publishing, New York
Lardon A (2012) The γ-core in Cournot oligopoly TU-games with capacity constraints. Theor Decis 72:387–411
Maskin E (2003) Bargaining, coalitions, and externalities, Working paper, Institute for Advanced Study
Perry MK, Porter RH (1985) Oligopoly and the incentive for horizontal merger. Am Econ Rev 75:219–227
Rajan R (1989) Endogenous coalition formation in cooperative oligopolies. Int Econ Rev 30:863–876
Ray D, Vohra R (1997) Equilibrium binding agreements. J Econ Theory 73:30–78
Radner R (2001) On the core of a cartel. In: Debreu G, Neuefeind W, Trockel W (eds) Economic essays. Springer, Berlin, pp 315–331
Salant SW, Switzer S, Reynolds RJ (1983) Losses from horizontal merger: the effects of an exogenous change in industry structure on Cournot–Nash equilibrium. Q J Econ 98:185–199
Shubik M (1975) Oligopoly theory, communication and information. Am Econ Rev 65(2):280–283
Stamatopoulos G (2016) The core of aggregative cooperative games with externalities. BE J Theor Econ 16:389–410
Thrall R, Lucas W (1963) N-person games in partition function form. Nav Res Logist Q 10:281–298
Yi SS (1997) Stable coalition structures with externalities. Games Econ Behav 20:201–237
Yong J (2004) Horizontal monopolization via alliances, or why a conspiracy to monopolize is harder than it appears. MIAESR University of Melbourne, Melbourne
Zhao J (1996) The hybrid solutions of an *n*-person game. Games Econ Behav 4:145–160
Zhao J (1999) A β-core existence result and its application to oligopoly markets. Games Econ Behav 27:153–168